**Problem:**
Show \(3 = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \ldots}}} \). 

First, one must figure out how to make sense out of “infinitely many square roots”. The natural way to do it is to recast the problem in terms of limits; we do this as follows:

**Problem Restated:** Show

\[
3 = \lim_{n \to \infty} \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \ldots\sqrt{1 + (n-1)\sqrt{1 + n}}} \}} 
\]

**Solution 1**

Note that

\[
3 = \sqrt{1 + 2\cdot 4} = \sqrt{1 + 2\sqrt{16}} = \sqrt{1 + 2\sqrt{1 + 3\sqrt{25}}} = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{36}}} \} = 3
\]

Based on this, one conjectures the relation

\[
3 = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \ldots\sqrt{1 + n\sqrt{(n+2)^2}}} \}}
\]

and this is seen to be true by induction.

From this, one sees immediately that

\[
3 \geq \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \ldots\sqrt{1 + (n-1)\sqrt{1 + n}}} \}}
\]

for all values of \(n\).

Now observe that for any \(a > 1\),

\[
\sqrt{1 + an} \leq \sqrt{a\sqrt{1 + n}}
\]

Thus, we see the following pattern emerge from (1.3):

\[
3 = \sqrt{1 + 2\cdot(2 + 2)} \geq (2 + 2)^{\frac{1}{2}} \sqrt{1 + 2} \quad \text{(think } n = 2) \\
3 = \sqrt{1 + 2\sqrt{1 + 3\cdot(3 + 2)}} \geq \sqrt{1 + 2(3 + 2)^{\frac{1}{2}} \sqrt{1 + 3}} \geq (3 + 2)^{\frac{1}{4}} \sqrt{1 + 2\sqrt{1 + 3}} \quad \text{(think } n = 3) \\
\vdots \\
etc.

so that we have, for any \( n \), that

\[
3 \leq (n + 2)^{2^{-n}} \cdot \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \ldots \sqrt{1 + (n-1)\sqrt{1 + n}}}}}
\]

Taken together, (0.4) and (0.7) establish the validity (0.1)

**Solution 2**

Let us define

\[
f(x) = \sqrt{1 + x} \sqrt{1 + (x+1)} \sqrt{1 + (x+2)} \ldots
\]

where the domain of definition is all \( x \) where the above makes sense.

Formal calculations tell us that

\[
[f(x)]^2 = 1 + xf(x+1),
\]

and we can observe that

\[
f(x) = x + 1
\]

satisfies (0.9).

We must still establish that the functions in (0.10) and (0.8) are identical. We do this by proving some inequalities that try to take advantage of the fact that (0.10) satisfies the functional equation (0.9). First, we have the following lower bound

\[
f(x) \geq \sqrt{x} \sqrt{x} \sqrt{x} \ldots = x \geq \frac{1}{2}(x+1)
\]

Secondly, we obtain an upper bound for \( f(x) \) in terms of \( x+1 \):

\[
f(x) \leq \sqrt{(x+1)\sqrt{(x+2)\sqrt{(x+3)\ldots}}}
\]

\[
\leq \sqrt{(x+1)\sqrt{2(x+1)\sqrt{4(x+1)\ldots}}}
\]

\[
= (x+1)\sqrt{2\sqrt{4\sqrt{\ldots}}}
\]

\[
= 2(x+1)
\]

Summarizing, we have

\[
\frac{1}{2}(x+1) \leq f(x) \leq 2(x+1),
\]

hence

\[
\frac{1}{2}(x+2) \leq f(x+1) \leq 2(x+2).
\]
An immediate consequence of (0.9) is

\[
\frac{1}{2} + x f(x+1) \leq [f(x)]^2 \leq 2 + x f(x+1).
\]

When we combine (0.15) and (0.14), we deduce

\[
\frac{1}{2} + \frac{1}{2} x(x+2) \leq [f(x)]^2 \leq 2 + 2x(x+2)
\]

\[
\frac{1}{2}(1 + x(x+2)) \leq [f(x)]^2 \leq 2(1 + x(x+2))
\]

\[
\frac{1}{2} (x+1)^2 \leq [f(x)]^2 \leq 2(x+1)^2
\]

\[
\frac{\sqrt{1}}{\sqrt{2}} \cdot (x+1) \leq f(x) \leq \sqrt{2} \cdot (x+1)
\]

Iterating the above, we obtain

\[
\left(\frac{1}{2}\right)^{\frac{1}{2k}} (x+1) \leq f(x) \leq 2^{\frac{1}{2k}} (x+1).
\]

Letting \( k \) approach infinity, we deduce

\[
x + 1 \leq f(x) \leq x + 1
\]

hence

\[
f(x) = x + 1
\]

and we finish upon setting \( x = 2 \).