Erratum to
“Fields of surreal numbers and exponentiation”
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by

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Lemma 4.5 in [2] is false. The correct result is the Lemma below. We use the following conventions and notations: $\Gamma$ is an ordered abelian group, $S \subseteq \Gamma$; we let $[S] := \{s_1 + \ldots + s_k : k \in \mathbb{N}, s_1, \ldots, s_k \in S\}$ be the additive monoid generated by $S$ in $\Gamma$; for $a \in \Gamma$, put $S^{<a} := \{s \in S : s < a\}$ and define $S^{\leq a}$ and $S^{\geq a}$ similarly; if $S$ is well-ordered, we let $o(S)$ be its ordinal. Also $\alpha, \lambda, \mu$ are ordinals, and sums and products of ordinals are their natural sums and natural products.

**Lemma.** Suppose $S \subseteq \Gamma^{\geq 0}$ is well-ordered with $o(S) \leq \mu$. Then $[S]$ is well-ordered with $o([S]) \leq \omega^\mu$.

Lemma 4.5 in [2] claims the sharper bound $o([S]) \leq \omega^\mu$. We will see below that this is correct if $\mu < \varepsilon_0$, but incorrect for $\mu = \varepsilon_0$.

Replacing Lemma 4.5 in [2] by the lemma above does not affect any of the main results of [2] but leads to minor changes in some proofs:

1. In the proof of Lemma 4.6, replace “$\omega^\alpha$” by “$\omega^{\alpha \tau}$” and “$\omega^\tau$” by “$\omega^{\alpha \omega}$”.

2. Lemma 4.10: in its statement and proof, replace “$\omega^{(\omega+\tau)\mu}$” by “$\omega^{(\alpha+\mu)}$”, and in its proof replace “$\omega^{\tau \mu}$” by “$\omega^{\alpha \omega \mu}$”.

3. In the proofs of Proposition 4.11, Lemma 5.2 and Lemma 5.4, replace “$\omega + 1$” (occurring as a factor in some exponents) by “$\omega 2$”, and “$2 \omega + 2$” by “$\omega 2$”.

**Proof of Lemma.** We proceed by induction on $\mu$. The lemma holds trivially for $\mu = 0$ ($S = \emptyset$) and $\mu = 1$, so let $\mu > 1$, and assume inductively that the desired result holds for smaller values.

[1]
Case 1: $\mu$ is not additive. This means that $\mu = \mu_1 + \mu_2$ for ordinals $\mu_1, \mu_2 < \mu$. Then $S = S_1 \cup S_2$ with $o(S_1) \leq \mu_1$ and $o(S_2) \leq \mu_2$. Hence $[S] = [S_1] + [S_2]$, so
$$o([S]) \leq o([S_1]) \cdot o([S_2]) \leq \omega^{\omega^{\mu_1}} \cdot \omega^{\omega^{\mu_2}} = \omega^{\omega^{\mu}}.$$

Case 2: $\mu$ is additive. Then $\mu = \omega^\lambda$, $\lambda > 0$. Let $0 < a \in S$, $0 < n \in \mathbb{N}$. It suffices to show that then $[S]^{\leq na} < \omega^{\omega^\mu}$, since the elements $na$ are cofinal in $[S]$. Note that $[S]^{\leq na} \subseteq [S^{\leq a}] + (S \cup \{0\}) + \ldots + (S \cup \{0\})$ where there are $n$ terms $S \cup \{0\}$. Hence, with $o(S^{\leq a}) = \alpha < \mu$, and using the fact that $o(S \cup \{0\}) \leq \mu = \omega^\lambda$, we obtain
$$o([S]^{\leq na}) \leq \omega^{\omega^\alpha \omega^\lambda \ldots \omega^\lambda} = \omega^{\omega^\alpha + n \lambda}.$$
Thus it remains to show that $\omega^\alpha + n\lambda < \mu$. To this end we write $\alpha$ in Cantor normal form as $\alpha = \omega^{\alpha_1} n_1 + \ldots + \omega^{\alpha_k} n_k$ with $\alpha > \alpha_1 > \ldots > \alpha_k$ and positive integers $n_1, \ldots, n_k$. Then the Cantor normal form of $\omega^\alpha$ has leading term $\omega^{\alpha_1 + 1} n_1$, so $\omega^\alpha \leq \omega^\lambda(n_1 + 1) = (n_1 + 1)\mu$. Hence $\omega^\alpha + n\lambda \leq (n_1 + 1)\mu + n\mu = (n_1 + 1 + n)\mu < \mu$.

In trying to carry out a similar inductive proof with the bound $\omega^\mu$ (instead of $\omega^{\omega^\mu}$), case 1 presents no problem, but case 2 leads to the inequality $\alpha + n\lambda < \mu$ (instead of $\omega^\alpha + n\lambda < \mu$). This inequality holds for $\lambda < \mu$, since $\mu$ is additive, but it fails when $\lambda = \mu$, that is, when $\mu$ is an $\varepsilon_0$-number. We conclude that the original lemma 4.5 in [2] holds for $\mu < \varepsilon_0$.

Lemma 4.5 fails for $\mu = \varepsilon_0$: Let $I = \mathbb{R}$, the additive ordered group of real numbers, and take for $S$ a well-ordered subset of the open interval $(0, 1)$ with $o(S) = \varepsilon_0$. Then $S \subseteq [S]$ and $[S]$ has elements $\geq 1$, so $\varepsilon_0 < o([S])$. Thus $o([S]) > \omega^{\varepsilon_0} = \varepsilon_0$.

The Remark following lemma 4.5 is also incorrect. (It did not play any further role in [2].) First, the assumption “$S \subseteq K^{>0}$” in this Remark should be replaced by “$S \subseteq K^{>1}$”. Then a correct bound follows by noting that the semiring generated by $S$ equals the additive monoid generated by the multiplicative monoid generated by $S$. This multiplicative monoid has ordinal at most $\omega^{\omega^\mu}$ by our corrected lemma, and thus the semiring generated by $S$ has ordinal at most $\omega^{\omega^{\omega^\mu}}$, which equals $\omega^{\omega^{1+\mu}}$. The Remark gives instead the bound $\omega^{\omega^\mu}$. This last bound (with $S \subseteq K^{>1}$) is correct for $\mu < \varepsilon_0$ (by the valid part of Lemma 4.5), but incorrect for $\mu = \varepsilon_0$ (by the counterexample in the last paragraph).

Earlier results on $o([S])$ are by Carruth [1] and by Gonshor and Harkleroad [4].

We take this opportunity to point out that part (3) of Lemma 4.2 in [2] is immediate from Theorem 5.12 of [3].
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References


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