INTRODUCTION

“Bridging the gap between the domains of discreteness and of continuity, or between arithmetic and geometry, is a central, presumably even the central problem of the foundations of mathematics” [1973, p. 211]. So wrote Abraham Fraenkel, Yehoshua Bar-Hillel and Azriel Levy in their mathematico-philosophical classic *Foundations of Set Theory*. “To understand the nature of the problem”, they went on to add,

one should stress the fundamental difference between the discrete, qualitative, individual nature of number in the “combinatorial” domain of counting (arithmetic) and the continuous, quantitative, homogeneous nature of the points of space (or of time) in the “analytical” domain of measuring (geometry). Every integer differs from every other in characteristic individual properties comparable to the differences between human beings, while the continuum appears as an amorphous pulp of points which display little individuality.

Bridging the gap between these two heterogeneous domains is not only the central but also the oldest problem in the foundations of mathematics and in the related philosophical fields. [1973, p. 212]

Cantor and Dedekind of course believed they had bridged the gap with the creation of their arithmetico-set theoretic continuum $\mathbb{R}$ of real numbers, and for roughly a century now it has been one of the central tenets of standard mathematical philosophy that indeed they had. In accordance with this view the geometric linear continuum is assumed to be isomorphic with the arithmetic continuum, the axioms of geometry being
so selected to ensure this would be the case. In honor of Cantor and Dedekind, who first proposed this mathematico-philosophical thesis, the transference of \( \mathbb{R} \)’s purported continuity to the continuity of the Euclidean straight line has come to be called the *Cantor-Dedekind axiom*. Given the Archimedean nature of the real number system,\(^1\) once this axiom is adopted we have the classic result of standard mathematical philosophy that infinitesimals are superfluous to the analysis of the structure of a continuous straight line.

More than twenty years ago, however, we began to suspect that while the Cantor-Dedekind theory succeeds in bridging the gap between the domains of arithmetic and of classical Euclidean geometry, it only reveals a glimpse of a far richer theory of continua that not only allows for infinitesimals, but leads to a vast generalization of portions of Cantor’s theory of the infinite, a generalization that also provides a setting for Abraham Robinson’s nonstandard approach to analysis [1961] as well as for the profound and all too often overlooked non-Cantorian theories of the infinite (and infinitesimal) pioneered by Giuseppe Veronese [1891], Tullio Levi-Civita [1892; 1898], David Hilbert [1899] and Hans Hahn [1907] in connection with their work on non-Archimedean ordered algebraic and geometric systems, and by Paul du Bois-Reymond [1870-71; 1875; 1877; 1882], Otto Stolz [1883], Felix Hausdorff [1907; 1909] and G. H. Hardy [1910] in connection with their work on the rate of growth of real functions. Central to the theory is J. H. Conway’s ordered field of *surreal numbers* [1976; 2001], a system of numbers containing the reals and the ordinals as well as a great many less familiar numbers including \( -\omega , \omega/2 , 1/\omega , \sqrt{\omega} \) and \( \omega - \pi \) to name only a few. Indeed, this particular number system, which Conway calls \( No \), is so remarkably inclusive that, subject to the proviso that numbers—construed here as members of ordered number fields\(^2\)—be individually definable in terms

\(^1\) In the case of an ordered field \( A \) (such as the ordered field of real numbers), the Archimedean axiom asserts: for all \( a,b \in A \), if \( 0 < a < b \), there is a positive integer \( n \) such that \( na > b \).

\(^2\) By “ordered number field”, we simply mean an ordered field whose elements are customarily called “numbers”.

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of sets of von Neumann-Bernays-Gödel set theory with Global Choice (NBG),\(^3\) it may be said to contain “All Numbers Great and Small.”\(^4\)

However, in addition to its inclusive structure as an ordered field, \(\mathbb{N}o\) has a rich hierarchical structure that emerges from the recursive clauses in terms of which it is defined. From the standpoint of Conway’s construction, this algebraico-tree-theoretic structure, or *simplicity hierarchy*, as we have called it [Ehrlich 1994], depends upon \(\mathbb{N}o\)’s *implicit* structure as a lexicographically ordered binary tree and arises from the fact that the sums and products of any two members of the tree are the simplest possible elements of the tree consistent with \(\mathbb{N}o\)’s structure as an ordered group and an ordered field, respectively, it being understood that \(x\) is *simpler than* \(y\) just in case \(x\) is a predecessor of \(y\) in the tree. In [Ehrlich 1994] the just-described simplicity hierarchy was brought to the fore and made a part of an algebraico-tree-theoretic definition of \(\mathbb{N}o\), and in [Ehrlich 2001] a novel class of structures whose properties generalize those of \(\mathbb{N}o\) so construed was introduced and some of the relations that exist between \(\mathbb{N}o\) and this more general class of *s-hierarchical ordered structures*, as we call them, was explored. Among the

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\(^3\) NBG is a conservative extension of ZFC. This being the case, its sets essentially coincide with those of standard set theory, i.e. those of ZFC. Unlike ZFC, however, NBG admits sets as well as *proper classes*. Throughout the text we follow the standard practice of referring to sets and proper classes collectively as *classes*. In virtue of the Axiom of Global Choice, all proper classes have the same ‘cardinality’ in NBG, the ‘cardinality’ of the class of all ordinals. For a discussion of NBG and its relation to ZFC, see [Fraenkel, Bar-Hillel and Levy, 1973].

\(^4\) The relation between the system of surreal numbers and the aforementioned number systems has been developed in a number of talks given by the author over the years (cf. [Ehrlich 2002 and 2004]), are touched on in [Ehrlich 2005], and will be treated in detail in a forthcoming work. For references and historical overviews of the non-Cantorian theories of the infinite and infinitesimal referred to above, see [Ehrlich 1995 and 2006]. For mathematical treatments of the relation between the surreal numbers and the work of Hahn, see [Ehrlich 1988 and 2001], and for mathematical treatments of the relation between the surreal numbers and the work of Cantor, see [Ehrlich 2001 and 1994]. Some of the latter work is discussed in Sections 3 and 10 below.

In addition to the relation between the system of surreal numbers and the aforementioned number systems, there is an intriguing relation between the system of surreal numbers and C. S. Peirce’s ideas on linear continua. For dicussions of this relation, see [Ehrlich 2005a; and, especially, Ehrlich forthcoming].
remarkable results that emerged from the latter investigation is that much as the surreal numbers emerge from the empty set of surreal numbers by means of a transfinite recursion that provides an unfolding of the entire spectrum of numbers great and small (modulo the aforementioned provisos), the recursive process of defining No’s arithmetic in turn provides an unfolding of the entire spectrum of ordered number fields in such a way that an isomorphic copy of each such system either emerges as an initial subtree of No or is contained in a theoretically distinguished instance of such a system that is.

In a number of works [Ehrlich 1987; 1989a; 1992; 1994, p. xxviii; 2005, p. 508], we have suggested that whereas the real number system should merely be regarded as constituting an arithmetic continuum modulo the Archimedean axiom, the system of surreal numbers may be regarded as a sort of absolute arithmetic continuum (modulo NBG). In the present paper, as part of a more general exposition of the purported absolute arithmetic continuum, we will outline some of the properties of the system of surreal numbers that we believe lend credence to this mathematico-philosophical thesis. In doing so, we hope to provide a gateway to some of the leading ideas of the theory that have thus far not found their way out of the theorem-proving literature. Following the above, we will draw attention to how properties of the system of surreal numbers considered as an s-hierarchical ordered algebraic structure can be appealed to in conjunction with classical relations between ordered algebraic and geometric systems to help resolve, for the surreal case, one of the purported difficulties that has historically lied at the heart of attempts to bridge the gap between the domains of number and of geometrical magnitude. That is, inspired by the challenge implicit in the remarks of Fraenkel, Bar-Hillel and Levy that don the title page of our text, we will attempt to explain how it is indeed possible that despite the fact that every surreal number differs from every other in characteristic individual properties comparable to the differences between human beings, absolute (elementary Euclidean) geometrical continua, which are modeled by the Cartesian space over the ordered field of surreal numbers together with its linear substructures, appear as
amorphous pulps of points which display little individuality. Among the results that are central to our resolution is that despite the fact that \( \text{No considered as an } s\text{-hierarchical ordered structure admits only the identity automorphism, the Cartesian space over the ordered field of surreal numbers admits a vast array of nontrivial automorphisms in which any given point may be associated with any other given point.} \) Indeed, as the following discussion will show, it is the paucity of isomorphisms from \( \text{No} \) (considered as an \( s\)-hierarchical ordered structure) onto isomorphic copies of itself together with other significant consequences of its \( s\)-hierarchical ordered structure that permits surreal numbers, unlike the points of an absolute geometrical continuum, to be assigned canonical proper names that are reflections of their characteristic individual properties.\(^6\)

1. THE SURREAL NUMBER TREE: A PRELUDE

\[^5\] While our discussion will be restricted to plane geometry and to the linear substructures thereof, our analysis readily generalizes to three-dimensional and, indeed, \( n\)-dimensional absolutely continuous elementary Euclidean geometry more generally.

\[^6\] Needless to say, authors, like Brouwer, who believe that a geometrical continuum cannot be represented by standard set-theoretic means will find that our remarks regarding “bridging the gap” completely miss the mark. We hasten to emphasize, however, that our remarks are not directed to such authors but rather to readers who, in accordance with standard mathematical practice, believe that a geometrical continuum can indeed be so represented. As is well known, there are a variety of ways this can be done, the most familiar being the classical approach of Hilbert [1899; 1971] using a many-sorted relational structure having separate universes for points, lines and planes. For our purpose, however, it proves to be more convenient to employ the representation due to Tarski [1959] in which points are the sole individuals, and lines and planes are treated as relational structures whose universes are particular classes of points.

As was intimated above, throughout the discussion the underlying set theory is assumed to be NBG. Some might object on philosophical grounds that the points that comprise the universe of a geometrical space should not be represented by classes, which are the sole entities of NBG. Without addressing the merits of this view, we note that (if so desired) the entire discussion could be carried out in a suitable variation of NBG that admits a proper class of \( \text{atoms} \) or \( \text{urelements} \) (called points) without impacting any of the relevant results. Heretofore, variations of NBG that admit atoms have been used primarily to establish independence results. For a discussion of such a variant of NBG, see [Rubin and Rubin 1985, pp. xi-xxvii].
In Conway’s development of the theory of surreal numbers the system of surreal numbers has the structure of an ordered field, the field operations like the surreal numbers themselves being defined by transfinite induction. Conway implicitly shows that the inductively defined ordered class of surreal numbers can be given the structure of a lexicographically ordered full binary tree, but this structure plays a limited role in Conway’s treatment.\(^7\) As in several of the classical constructions of real numbers, surreal numbers for Conway are formally construed as equivalence classes. In accordance with Conway’s construction, each surreal number is in fact an equivalence class consisting of an entire proper class of equivalent representations, which makes it impossible to collect the proper class of all surreal numbers thus construed into a legal class of NBG. If one wishes, one can carry out Conway’s construction in the equiconsistent set theory of Ackermann (cf. [Ehrlich 1994, p. 246]). In NBG, on the other hand, the numbers must be ‘cut down to size’; this can be done most easily by identifying each number with a canonical representative of each of Conway’s equivalence classes, or by identifying each number with a canonical subset of each class.\(^8\) For the purpose of formalization, Conway proposes the latter approach, making use of the set-theoretic rank function [1976, p. 65]. By contrast, we have introduced two alternative constructions that employ canonical representatives: one is based on cuts due to Cuesta Dutari [1954] that generalize the familiar cuts of Dedekind [Ehrlich 1988; Alling, and Ehrlich 1986; 1987], and the other is

\(^7\) What does play a role on occasion in Conway’s development is a \textit{birthday} function that maps each surreal number to the level of recursion at which it is created (as well as a related notion of \textit{simplicity}). For remarks on Conway’s birthday map and its relation to the development of surreal numbers presented here, see [Ehrlich 1994, p. 257: Note 1] and [Ehrlich 2001, p. 1232: Note 2].

\(^8\) I am assuming here that surreal numbers are \textit{games} in Conway’s sense or equivalence classes of such games (see Section 3 above). As Conway mentioned [1976, p. 65], the difficulties of formalizing his approach in NBG can be sidestepped by abandoning this assumption and employing sign-expansions of surreal numbers (see Section 4 above) as the surreal numbers themselves (cf. [Gonshor 1986] and [Ehrlich 2001]).
a generalization of the von Neumann ordinal construction [Ehrlich 1994; 2002a]. In the present discussion, following a review of some of the basic definitions from the theory of lexicographically ordered binary trees, we will introduce the surreal numbers via the latter approach.

2. LEXICOGRAPHICALLY ORDERED BINARY TREES:

PRELIMINARY DEFINITIONS

A tree \( \langle A, <_A \rangle \) is a partially ordered class such that for each \( x \in A \), the class \( \{ y \in A : y <_A x \} \) of predecessors of \( x \), written \( pr_A(x) \), is a set well-ordered by \( <_A \). A maximal subclass of \( A \) well-ordered by \( <_A \) is called a branch of the tree. Given any two distinct elements \( x \) and \( y \) of \( A \), precisely one of the following is the case: either \( x \) is a predecessor of \( y \) (i.e. \( x <_A y \)), \( y \) is a predecessor of \( x \) (i.e. \( y <_A x \)) or neither \( x <_A y \) nor \( y <_A x \). In the latter case, \( x \) is said to be incomparable with \( y \). Thus, for distinct elements \( x \) and \( y \) of \( A \), \( x \) is incomparable with \( y \) if and only if \( x \) and \( y \) lie on different branches of the tree. An initial subtree of \( \langle A, <_A \rangle \) is a subclass \( A' \) of \( A \) with the order inherited from \( <_A \) such that for each \( x \in A' \), the set of predecessors of \( x \) in \( A' \) coincides with the set of predecessors of \( x \) in \( A \). The tree-rank of \( x \in A \), written \( \rho_A(x) \), is the ordinal corresponding to the well-ordered set of predecessors of \( x \); the \( \alpha \)th level of \( A \) is the set of all members of the tree having tree-rank \( \alpha \); and a root of \( A \) is a member of the zeroth level. If \( x, y \in A \), then \( y \) is said to be an immediate successor of \( x \) if \( x <_A y \) and \( \rho_A(y) = \rho_A(x) + 1 \); and if \( (x_\alpha)_{\alpha<\beta} \) is a chain in \( A \) (i.e., a subclass of \( A \) well-
ordered by $<_A$), then $y$ is said to be an immediate successor of the chain if $x_\alpha <_A y$ for all $\alpha < \beta$ and $\rho_A(y)$ is the least ordinal greater than the tree-ranks of the members of the chain. The length of a chain $(x_\alpha)_{\alpha < \beta}$ in $A$ is the ordinal $\beta$.

A tree $\langle A, <_A \rangle$ is said to be binary if each member of $A$ has at most two immediate successors and every chain in $A$ of limit length has at most one immediate successor. If every member of $A$ has two immediate successors and every chain in $A$ of limit length (including the empty chain) has an immediate successor, then the binary tree $\langle A, <_A \rangle$ is said to be full. Since a full binary tree has a level for each ordinal, the universe $A$ of a full binary tree $\langle A, <_A \rangle$ is a proper class.

Using the Axiom of Global Choice (or simply the Axiom of Choice if $A$ is a set) a tree may be shown to be binary if and only if it is isomorphic to an initial subtree of the canonical full binary tree $\langle B, <_B \rangle$, where $B$ is the class of all sequences of +s and -s indexed over some ordinal and $x <_B y$ signifies that $x$ is a proper initial subsequence of $y$ (cf. [Drake 1974, p. 216]).

By an ordered tree $\langle A, <,<_A \rangle$, we mean a tree $\langle A, <_A \rangle$ whose universe is totally ordered by $<$. Thus, an ordered tree $\langle A, <,<_A \rangle$ has two orderings: a partial ordering $<_A$ in virtue of which $\langle A, <_A \rangle$ is a tree and a total ordering $<$ in virtue of which $\langle A, < \rangle$ is an ordered class. As is well known, $\langle B, <_B \rangle$ can be totally ordered (lexicographically) in accordance with the definition:

$$(x_\alpha)_{\alpha < \mu} <_{\text{lex}(B)} (y_\alpha)_{\alpha < \sigma} \text{ if and only if } x_\beta = y_\beta \text{ for all } \beta < \text{ some } \delta, \text{ but } x_\delta < y_\delta,$$

it being understood that $- < \text{undefined} < +$.

The resulting structure $\langle B, <_{\text{lex}(B)}, <_B \rangle$ is called the lexicographically ordered canonical full binary tree.
Figures 1-3 (below) exhibit levels 0-3 of $\langle B, \leq_{\text{lex}(B)}, <_n \rangle$, the level of the tree being indicated in Figure 1 on the right. In Figure 2 the members of the tree that are connected to a given member by a strictly descending sequences of dots are the predecessors of the given member. And in Figure 3 the ordering of the positions of the projected sequences on the line indicate the total ordering of the members of levels 0-3 in accordance with the lexicographical ordering.
In Section 4 we will explore the relation between the surreal number tree and $\langle B, \langle \text{lex}(B), <_B \rangle \rangle$. For the time being, however, we restrict our attention to the former.

3. THE SURREAL NUMBER TREE

Von Neumann defines an ordinal as a transitive set that is well-ordered by the membership relation. As a result, for von Neumann, an ordinal emerges as the set of all of its predecessors in the ‘long’ though rather trivial binary tree $\langle \text{Ord}, \varepsilon \rangle$ of all ordinals. So, for example, 0 is identified with $\emptyset$, 1 is identified with $\{\emptyset\} = \{0\}$, 2 is identified with $\{\emptyset, \{\emptyset\}\} = \{0,1\}$ and so on. In our construction of surreal numbers, which generalizes von Neumann’s ordinal construction, each surreal number $x$ emerges as an ordered pair $(L_x, R_x)$ of sets of surreal numbers that are predecessors of $x$. Although $L_x$ and $R_x$ are defined independently of the total ordering that will be imposed on the number tree, they ultimately coincide with the sets of all predecessors of $x$ that are less than $x$ and greater than $x$, respectively.

$\langle \text{Ord}, \varepsilon \rangle$ is not usually described as a binary tree since its familiar structure as a well-ordered class is indistinguishable from its structure as a tree. In the theory of surreal numbers, however, one must distinguish between its binary tree structure, on the one hand, and its structure as a totally ordered class, on the other hand.

Inspired by von Neumann’s aforementioned definition of an ordinal, in [Ehrlich
we introduced an analogous explicit definition of a surreal number. Here, however, we will introduce the surreal numbers inductively [Ehrlich 1994: Appendix III]. The particular inductive approach we will employ requires one to have a surreal number \((L, R)\) at hand before it can be identified as a surreal number. To ensure the availability of the requisite class of set-theoretical entities, we follow Conway in first introducing the class of \textit{games}.

\textbf{CONSTRUCTION OF GAMES}

If \(L\) and \(R\) are any two sets of games, then there is a game \((L, R)\). All games are constructed in this way.

It is not difficult to see that this construction leads to an entire proper class of games beginning with the game \((\emptyset, \emptyset)\), where \(\emptyset\) is the empty set (of games). The closure condition “All games are constructed in this way” simply ensures that nothing is a game that doesn’t arise in the specified way. To inductively extract the surreal numbers from the class of games we make use of the following terminology from [Ehrlich 2002a].

\textbf{PRELIMINARY DEFINITIONS}

A game \(x\) is said to be \textit{simpler than} a game \(y = (L_y, R_y)\), written \(x <_s y\), if \(x \in L_y\) or \(x \in R_y\); a chain of games (i.e., a class of games totally ordered by \(<_s\)) is said to be \textit{ancestral} if it is closed under the simpler than relation, i.e., \(x\) is a member of the chain whenever \(y\) is a member of the chain and \(x <_s y\); and a partition \(L, R\) of an ancestral chain of games is said to be \textit{orderly}, if \(L \supseteq L_x\) and \(R_x \subseteq R\) for each element \(x = (L_x, R_x)\) of the chain.
The members of the class $No$ of surreal numbers are now identified by means of the following simple construction in which the clause “there is a surreal number $(L, R)$” signifies that “the game $(L, R)$ is a surreal number.”

**CONSTRUCTION OF SURREAL NUMBERS**

If $L, R$ is an orderly partition of an ancestral chain of surreal numbers, then there is a surreal number $(L, R)$. All surreal numbers are constructed in this way.

At this point it is not difficult to show that $\langle No, <_s \rangle$ is a full binary tree where the definition of the simpler than relation for surreal numbers is inherited from the definition for games. For this purpose, however, it is convenient to have available the ordinals. If one wishes, one could avail oneself of the von Neumann ordinals, which are already at hand. On the other hand, if one wants to develop the theory of ordinals within the theory of surreal numbers, as we intend to do, before proving the above theorem one must first identify “our” ordinals.

**ISOLATION OF THE ORDINALS**

A surreal number $x = (L_x, R_x)$ will be said to be an ordinal if $R_x = \emptyset$. By $On$ we mean the class of ordinals so defined. For all ordinals $x$ and $y$, $x$ will be said to be less than $y$, written $x <_{On} y$, if $L_x \subseteq L_y$.

In accordance with the above definition, $(\emptyset, \emptyset)$ is an ordinal, $((\emptyset, \emptyset), \emptyset)$ is an ordinal, $\left[ (\emptyset, \emptyset), \left( (\emptyset, \emptyset), \emptyset \right) \right], \emptyset$ is an ordinal, and so on. In fact, the just-cited ordinals are eventually identified with the finite ordinals $0 = (\emptyset, \emptyset)$, $1 = \left( \left\{ 0 \right\}, \emptyset \right)$ and $2 = \left( \left\{ 0, 1 \right\}, \emptyset \right)$, respectively.

That the ordered class $\langle On, <_{On} \rangle$ of ordinals has all of the requisite properties possessed by any of the more familiar constructs so-called follows from
Proposition 1. There is a one-to-one order preserving correspondence between the ordered class \( \langle \text{On},<_{\text{On}} \rangle \) and the ordered class \( \langle \text{Ord},\in \rangle \) of von Neumann ordinals.

Using the ordered class of ordinals so defined, the structure of the surreal numbers that emerge vis-à-vis the above construction may be characterized as follows.

Theorem 1 [Ehrlich 1994]. \((\emptyset,\emptyset)\) is a surreal number; if \( x = (L_x,R_x) \) is a surreal number, then \((L_x,\{x\} \cup R_x)\) and \((L_x \cup \{x\},R_x)\) are surreal numbers; moreover, if \( \{x_\alpha\}_{\alpha < \beta} \) is a chain of surreal numbers (ordered by \(<_s\)) of infinite limit length \( \beta \), then \((\bigcup_{\alpha < \beta} L_{x_\alpha}, \bigcup_{\alpha < \beta} R_{x_\alpha})\) is a surreal number. Nothing is a surreal number except in virtue of the above.\(^{10}\)

The reader will note that \((L_x,\{x\} \cup R_x)\) and \((L_x \cup \{x\},R_x)\) are the immediate successors in \( \langle \text{No},<_s \rangle \) of the surreal number \( x \), and that \((\bigcup_{\alpha < \beta} L_{x_\alpha}, \bigcup_{\alpha < \beta} R_{x_\alpha})\) is the immediate successor in \( \langle \text{No},<_s \rangle \) of the chain \( x_\alpha \) of surreal numbers indexed over the infinite limit ordinal \( \beta \). In fact, as it turns out, \((L_x,\{x\} \cup R_x)\) is the immediate successor of \( x \) less than \( x \), \((L_x \cup \{x\},R_x)\) is the immediate successor of \( x \) greater than \( x \), and \((\bigcup_{\alpha < \beta} L_{x_\alpha}, \bigcup_{\alpha < \beta} R_{x_\alpha})\) is always greater than the members of \( \bigcup_{\alpha < \beta} L_{x_\alpha} \) and less than the members of \( \bigcup_{\alpha < \beta} R_{x_\alpha} \). To lend precision to these ideas and to ideas regarding \( \text{No} \) considered as a totally ordered class more generally, however, we require

**THE RULE OF ORDER**

For all surreal numbers \( x = (L_x,R_x) \) and \( y = (L_y,R_y) \), \( x < y \) if and only if \( x \in L_y \) or \( y \in R_x \) or \( R_x \cap L_y \neq \emptyset \).

\(^{10}\) The reader will notice that if one has no interest in developing the theory of ordinals within the theory of surreal numbers itself, one may take the statement of Theorem 1 to be an inductive definition of the surreal numbers. If this course is followed, there is no need to first introduce the class of games.
The reader will note that the three cases that comprise the rule of order correspond to the three mutually exclusive and collectively exhaustive relations that two surreal numbers $x$ and $y$ may bear to one another when $x$ is less than $y$; namely, $x$ is simpler than $y$, $y$ is simpler than $x$, and $x$ is incomparable with $y$, respectively. Using this in conjunction with Theorem 1, one may prove

**Corollary 1** [Ehrlich 1994]. For each surreal number $x$,

$$x = (L_{s(x)}, R_{s(x)})$$

where $L_{s(x)} = \{a \in No : a <_x x \text{ and } a < x\}$ and $R_{s(x)} = \{a \in No : a <_x x \text{ and } x < a\}$.

Moreover, by appealing to Corollary 1 together with the definition of No’s ordinals and the definition of corresponding ordering thereof, we also have

**Corollary 2** [Ehrlich 1994]. No’s ordinals are the members of the rightmost branch of $\langle No, <, <_s \rangle$, the ordinal $\alpha$ being the member of tree-rank $\alpha$. As such, for all ordinals $\alpha$ and $\beta$, $\alpha <_{on} \beta$ if and only if $\alpha <_s \beta$ if and only if $\alpha < \beta$. Moreover, if $\alpha$ is an ordinal, then

$$\alpha = (L_{s(\alpha)}, \emptyset).$$

An automorphism is an isomorphism of a structure $\langle A, ... \rangle$ onto itself. Roughly speaking, if $f$ is an automorphism of $\langle A, ... \rangle$ in which $f(x) \neq x$ for some $x \in A$, then $f(x)$ and $x$ can play exactly the same roles as one another in isomorphic structures having the common universe $A$. As we emphasized in [Ehrlich 1994, p. 253] and the subsequent discussion will show, $\langle No, < \rangle$ has a vast array of automorphisms. However, as the following result makes clear, all but one fail to take into account from whence the numbers come.
Theorem 2 [Ehrlich 1994]. The identity map is the sole automorphism of \( \langle No, \prec \rangle \) that preserves predecessors. That is, an automorphism \( f \) of \( \langle No, \prec \rangle \) satisfies the condition \( pr_{No}(f(x)) = pr_{No}(x) = L_{s(x)} \cup R_{s(x)} \) for all \( x \in No \) if and only if \( f(x) = x \) for all \( x \in No \).\n
The identity map is likewise the only automorphism of \( \langle No, \prec \rangle \) that preserves tree-rank [Ehrlich 1994]. Moreover, the following consequence of Theorem 2 provides still another way of characterizing this map.

Corollary 3 [Ehrlich 1994]. \( \langle No, \prec, \prec_s \rangle \) has no automorphism other than the identity automorphism.

4. THE STRUCTURE OF \( \langle No, \prec, \prec_s \rangle \)

For each surreal number \( x \) there is a unique enumeration \( (x_\alpha)_{\alpha < \rho_{No}(x)} \) of the predecessors of \( x \) well-ordered by the simpler than relation. Henceforth, we refer to this enumeration as the genealogy of \( x \). By the sign-expansion of \( x \) we mean the sequence \( (g_\alpha(x))_{\alpha < \rho_{No}(x)} \) defined by the condition: for all \( \alpha < \rho_{No}(x) \)

\[
g_\alpha(x) = \begin{cases} +, & \text{if } x_\alpha \in L_{s(x)} \\ -, & \text{if } x_\alpha \in R_{s(x)} \end{cases}
\]

where \( x_\alpha \) is the \( \alpha \)th term in the genealogy of \( x \) (i.e., the predecessor of \( x \) having tree-rank \( \alpha \)).\(^{11} \)

It is not difficult to show that every surreal number has a sign-expansion; distinct surreal numbers have distinct sign-expansions; and every sequence of elements of \( \{+, -\} \) of length \( \beta \in On \) is the sign-expansion of some surreal number [Ehrlich 1994: Theorem 1.4]. Moreover, and more important, however, is

\(^{11} \) Our definition and treatment of sign-expansions differs considerably from Conway’s [1976, pp. 29-30]. In Conway’s treatment, sign-expansions are defined using the ordered additive structure of \( No \).
Theorem 3 [Ehrlich 1994]. \( \langle \mathbb{N}_0, <, \rangle \) and \( \langle B, <_{\text{lex}(B)}, <_B \rangle \) are isomorphic as ordered trees, the map that sends each surreal number to its sign-expansion being the unique such isomorphism.

While \( \langle B, <_{\text{lex}(B)}, <_B \rangle \) sheds important light on the structure of \( \langle \mathbb{N}_0, <, \rangle \), in the general theory of surreal numbers it is advantageous to have available a representation independent characterization of those ordered trees that are isomorphic to initial ordered subtrees of \( \langle B, <_{\text{lex}(B)}, <_B \rangle \) and, hence, of \( \langle \mathbb{N}_0, <, \rangle \). There are two particularly useful such characterizations that will be appealed to in the subsequent discussion, the first of which we take as the standard definition.

Definition 1 [Ehrlich 2001]. A lexicographically ordered binary tree is an ordered binary tree in which distinct members are incomparable if and only if they have a common predecessor lying between them.

A subclass \( I \) of an ordered class \( \langle A, < \rangle \) is said to be convex if every member of \( A \) that lies between two members of \( I \) is likewise a member of \( I \). The familiar intervals of an ordered class \( \langle A, < \rangle \) of the forms \( \{ x \in A: a \leq x \leq b \} \), \( \{ x \in A: a < x < b \} \), \( \{ x \in A: a \leq x < b \} \) and \( \{ x \in A: a < x \leq b \} \) are all convex subclasses. Besides these, however, there may be convex subclasses of \( \langle A, < \rangle \) without a greatest lower bound \( a \) or without a least upper bound \( b \). The ordered set of rational numbers of course contains many such convex subclasses, and non-Archimedean ordered fields invariably contain many such convex subclasses as well.

The idea of a convex subclass of an ordered class plays a leading role in the formulation of the second of the two representation independent characterizations of a lexicographically ordered binary tree alluded to above. Also important in this regard are
the following notational and terminological conventions that will be employed throughout the remainder of the paper.

If \( L \) and \( R \) are subclasses of an ordered class \( \langle A, \dot{<} \rangle \), then ‘\( L < R \)’ signifies that every member of \( L \) precedes every member of \( R \). In addition, if \( x \) and \( y \) are members of an ordered tree \( \langle A, \dot{<}, \dot{s} \rangle \), then (as in the case of surreal numbers) \( x <_{s} y \) will be read “\( x \) is *simpler than* \( y \)”); moreover, an element \( x \) of a nonempty subclass \( I \) of \( A \) will be said to be “*the simplest member*” of \( I \) if \( x <_{s} y \) for all \( y \in I - \{x\} \). Finally, mimicking the stipulations introduced in Corollary 1, by ‘\( L_{s}(x) \)’ we mean \( \{a \in A : a <_{s} x \text{ and } a < x\} \) and by ‘\( R_{s}(x) \)’ we mean \( \{a \in A : a <_{s} x \text{ and } x < a\} \).

The second of our representation independent characterizations of a lexicographically ordered binary tree is given by

**Theorem 4** [Ehrlich 2001]. \( \langle A, \dot{<}, \dot{s} \rangle \) is a lexicographically ordered binary tree if and only if \( \langle A, \dot{<}, \dot{s} \rangle \) is an ordered tree in which every nonempty convex subclass contains a simplest member, and for all \( x, y \in A \), \( x <_{s} y \) if and only if \( L_{s}(x) < \{y\} < R_{s}(x) \) and \( x \neq y \).

Now suppose \( L \) and \( R \) are subsets of a lexicographically ordered binary tree \( \langle A, \dot{<}, \dot{s} \rangle \) where \( L < R \). It follows from the just-stated theorem that the convex subclass of \( A \) consisting of all the members of \( A \) lying between \( L \) and \( R \) is either empty or contains a simplest member. If a simplest such member always exists, \( \langle A, \dot{<}, \dot{s} \rangle \) will be said to be complete. As the following theorem makes clear, the concept of a complete lexicographically ordered binary tree and the concept of a full lexicographically ordered binary tree provides us with complementary perspectives on the nature of \( No \).

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12 In fact, as was shown in [Ehrlich 2001: Theorems 1 and 2], the characterization of a lexicographically ordered binary tree stated in Theorem 4 can be obtained from the following apparently weaker condition: every nonempty convex subclass of \( A \) contains a simplest member, and for all \( x, y \in A \), \( L_{s}(x) < \{y\} < R_{s}(x) \) whenever \( x <_{s} y \).
Theorem 5 [Ehrlich 1994; 2001]. A lexicographically ordered binary tree is complete if and only if it is full if and only if it is isomorphic to \( \langle \text{No},<,\leq \rangle \).

5. ORDERED FIELDS WITH SIMPLICITY HIERARCHIES

Let \( \langle A, <, \leq \rangle \) be a lexicographically ordered binary tree. If \( x \) is the simplest member of \( A \) lying between two subsets \( L \) and \( R \) of \( A \) where \( L < R \), then we will denote \( x \) by \( \{L \mid R\}^A \) or, simply, by \( \{L \mid R\} \) when no ambiguity will arise. Moreover, applying Conway’s conventions [Conway 1976; 2001, p. 4] to our definition of \( \{L \mid R\} \), if \( x = \{L \mid R\} \) we write \( x^L \) for the typical member of \( L \) and \( x^R \) for the typical member of \( R \); for \( x \) itself we write \( \{x^L \mid x^R\} \); and \( x = \{a,b,c,\ldots \mid d,e,f,\ldots \} \) means that \( x = \{L \mid R\} \) where \( a,b,c,\ldots \) are the typical member of \( L \) and \( d,e,f,\ldots \) are the typical member of \( R \).

The following consequence of Theorem 4 is central to the subsequent development.

Proposition 2. If \( \langle A, <, \leq \rangle \) is a lexicographically ordered binary tree and \( x \in A \), then there are (possibly empty) subsets \( L \) and \( R \) of \( A \) such that \( L < R \) and for which \( x = \{L \mid R\} \); in particular, \( x \) is the simplest member of \( A \) lying between its predecessors on the left and its predecessors on the right, that is, \( x = \{L_{x_L} \mid R_{x_R}\} \).

In the following definition \( x^L, x^R, y^L \) and \( y^R \) are understood to be typical members of arbitrary sets \( L_x, R_x, L_y \) and \( R_y \), respectively, for which \( x = \{L_x \mid R_x\} \) and \( y = \{L_y \mid R_y\} \).

Definition 2 [Ehrlich 2001]. \( \langle A, +, <, \leq, 0, 1 \rangle \) is said to be an \textit{s-hierarchical ordered field} if (i) \( \langle A, +, <, \leq, 0, 1 \rangle \) is an ordered field; (ii) \( \langle A, <, \leq \rangle \) is a lexicographically ordered binary tree; and (iii) for all \( x, y \in A \),

\[
x + y = \{x^L + y, x + y^L \mid x^R + y, x + y^R\}
\]

and
\[
xy = \{x^L y + xy^L - x^L y^L, x^R y + xy^R - x^R y^R | x^L y + xy^R - x^L y^R, x^R y + xy^L - x^R y^L\}. 
\]

The reader will note that in virtue of the conventions introduced at the beginning of this section, the s-hierarchical constraint on sums in the above definition asserts that 
\[
x + y = \{L_{x+y} | R_{x+y}\} \text{ where } L_{x+y} \text{ is the set consisting of all sums of the form } x^L + y \text{ where } x^L \in L_x \text{ as well as all sums of the form } x + y^L \text{ where } y^L \in L_y, \text{ and } R_{x+y} \text{ is the set consisting of all sums of the form } x^R + y \text{ where } x^R \in R_x \text{ as well as all sums of the form } x + y^R \text{ where } y^R \in R_y. \text{ The s-hierarchical constraint on products is analogously understood.}
\]

Using the s-hierarchical constraints on sums and products, respectively, it is a simple matter to show that 
\[
0 = \{\emptyset | \emptyset\} = \{\} \text{ is the simplest element of, as well as the unique root in, an s-hierarchical ordered field, and that } 1 = \{\{(\emptyset) | \emptyset\}| \emptyset\} = \{0\} \text{ is the simplest positive element [Ehrlich 2001]. Moreover, it is a consequence of the following result that except for the signs of elements, the portion of the tree less than 0 is a mirror image of the portion of the tree greater than 0.}
\]

**Proposition 3** [Ehrlich 2001]. For each element \(x\) of an s-hierarchical ordered field
\[
-x = \{-x^R | -x^L\}. 
\]

Like the just-stated algebraico-tree-theoretic condition governing additive inverses, the algebraico-tree-theoretic constraints placed on sums and products of elements of an s-hierarchical ordered field are apt to appear rather cryptic to readers unfamiliar with the theory of surreal numbers. We will decipher their significance shortly. Before doing so, however, we note that a good deal of their motivation arises from the following straightforward tree-theoretic variation of a remarkable result due to Conway [1976].
Theorem 5 [Ehrlich 2001]. \( \langle \text{No}, +, \cdot, <, <, , 0, 1 \rangle \) is an \( s \)-hierarchical ordered field when \(+, −\) and \( \cdot \) are defined by recursion as follows where \( x^L, x^R, y^L \) and \( y^R \) are understood to be typical members of \( L_{s(x)} \), \( R_{s(x)} \), \( L_{s(y)} \) and \( R_{s(y)} \), respectively.

**Definition of** \( x + y \).

\[ x + y = \{ x^L + y, x + y^L \mid x^R + y, x + y^R \} . \]

**Definition of** \( −x \).\(^{13}\)

\[ −x = \{ −x^R \mid −x^L \} . \]

**Definition of** \( xy \).

\[ xy = \{ x^L y + xy^L − x^L y^L, x^L y + xy^R − x^L y^R \mid x^L y + xy^R − x^L y^R \} . \]

As our remarks preceding the statement of Theorem 5 suggest, the constraints on sums and products found in Definition 2 are tree-theoretic adaptations of the recursive definitions of \(+\) and \( \cdot \) employed by Conway in his construction of the ordered field of surreal numbers [1976, 2001, p. 5]. Appealing to ideas alluded to in the Introduction, the import of these constraints may be encapsulated thus.

Let \( x, y \in \text{A} \). Since \( x = \{ x^L \mid x^R \} \) and \( y = \{ y^L \mid y^R \} \), it follows that for all \( x^L, x^R, y^L \) and \( y^R \):

\[ (*) \quad x^L < x < x^R \text{ and } y^L < y < y^R. \]

\(^{13}\) The definition of \( −x \) given above (i.e. Conway’s definition) is not actually required for the definition of \( \text{No} \); one can show that the ordered semigroup that arises from \(+\) and \( < \) is a group by showing that \( −x \) is the surreal number whose sign-expansion is obtained from the sign-expansion of the surreal number \( x \) by replacing the \(+\)’s and \( −\)’s in the sign-expansion of \( x \) with \( −\)’s and \( +\)’s, respectively. Having done so, one can always obtain Conway’s formula via Proposition 3. On the other hand, given the elegance of Conway’s definition and the fact that subtraction is required for the definition of multiplication, we have chosen to follow Conway’s lead.
Furthermore, since $A$ is an ordered field (and, hence, an ordered additive group), it follows that $x^L + y < x + y < x^R + y$ and $x + y^L < x + y < x + y^R$ and, hence, that:

\[ (** ) \quad \{ x^L + y, x + y^L \} < \{ x + y \} < \{ x^R + y, x + y^R \} . \]

Therefore, since $x + y$ must lie between the two sets of members of $A$ specified in (**), clause (iii) of Definition 2 requires that $x + y$ be the simplest member of $A$ lying between those sets. Similarly, since, by clause (i) of Definition 2, $A$ is an ordered field, it follows from (*) that each of the differences $x - x^L, x^R - x, y - y^L, y^R - y$ is positive and, hence, so is each of the products $(x - x^L)(y - y^L), (x^R - x)(y^R - y), (x - x^L)(y^R - y)$ and $(x^R - x)(y - y^L)$. But by applying this together with the routine (high school) algebra of ordered fields to each of these products one obtains:

\[ (***) \quad \{ x^L y + xy^L - x^L y^L, x^R y + xy^R - x^R y^R \} < \{ xy \} < \{ x^L y + xy^R - x^L y^R, x^R y + xy^L - x^R y^L \} . \]

Therefore, since $xy$ must lie between the two sets of members of $A$ specified in (**), clause (iii) of Definition 2 requires that $xy$ be the simplest member of $A$ lying between those sets.

Given an ordered field $\langle A, +, :, <, 0, 1 \rangle$, it is possible to define a lexicographically ordered binary tree structure on its ordered class $\langle A, < \rangle$ of elements in infinitely many ways. That is, there are infinitely many distinct predecessor relations $<_A$ that can be defined on the members of $A$ in accordance with which $\langle A, <_A \rangle$ is a lexicographically ordered binary tree. In the case of $\langle \mathbb{N}_0, +, :, <, 0, 1 \rangle$, in particular, a lexicographically ordered binary tree structure can be imposed on its ordered class of elements in as many ways as there are ordinals. Moreover, while the constraints on sums and products dictated by (** and (***) respectively, will be satisfied in any of the resulting systems, one is largely free to assign the sum and product of any two members of the system to any position in the tree one pleases. If the resulting structure is to be an s-hierarchical ordered
field, however, this is not the case. Indeed, requiring that the sums and products of members of $A$ be the simplest possible elements of the tree consistent with (***) and (***), is tantamount to requiring that sums and products of members of $A$ have the lowest possible tree rank consistent with those constraints. In the case of an s-hierarchical ordered field like $\text{No}$, whose sums and products are defined by recursion, this implies that the sums and products of elements get defined just as soon as there is sufficient previously defined ordered algebraico-tree theoretic information to do so. As we mentioned in the Introduction and the subsequent sections of the paper will show, this has profound implications for the s-hierarchical ordered field of surreal numbers.

6. S-HIERARCHICAL MAPPINGS AND S-HIERARCHICAL MORPHISMS

There is a uniquely defined image of each rational number $r$ in every ordered field. Accordingly, in every ordered number field we can speak unambiguously of the rational number $r$. Moreover, if $A$ is an ordered subfield of the ordered field of real algebraic numbers (i.e., the ordered subfield of all real numbers that are solutions to polynomial equations with integer coefficients) and $r$ is an irrational member of $A$, such as $\sqrt{2}$, then every ordered field containing an isomorphic copy of $A$ will have a uniquely defined image of $r$ as well. However, there are no other members of ordered number fields for which this is true without qualification. Of course, every member $r$ of an Archimedean ordered field has a unique image in the ordered field of real numbers and, as such, within the confines of an Archimedean ordered number field one can speak unambiguously of the real number $r$ whether it be a rational number, a real algebraic number that is not rational, or a transcendental number, i.e., an irrational number, such as $\pi$ or $e$, that is not a real algebraic number. However, if $A$ is an Archimedean ordered field containing a transcendental number $r$, then no non-Archimedean ordered field containing an isomorphic copy of $A$ will have a uniquely defined image of $r$. That is,
there will always be isomorphisms \( f \) and \( g \) from \( A \) into the non-Archimedean ordered field in question for which \( f(r) \neq g(r) \). In fact, if \( A \) is an arbitrary ordered number field, whose universe is a set, and \( r \) a member of \( A \) that is not one of \( A \)’s real algebraic numbers, there is an entire proper class of isomorphisms of \( A \) into \( No \) for which no two of them send \( r \) to the same surreal number!

Happily, however, as the remainder of this and the subsequent section will show, not only does the problem of the absence of uniquely defined images fail to arise for members of s-hierarchical ordered number fields, but every such element has a uniquely defined image in \( \langle No,+,\cdot,\prec,\preceq,0,1 \rangle \). In saying as much, we are alluding to the fact that the isomorphisms appropriate for s-hierarchical ordered fields are more stringent than those appropriate for ordered fields more generally. It is to these \emph{s-hierarchical morphisms}, as we call them, that we now turn.

**Definition 3** [Ehrlich 2001]. A mapping \( f:A \rightarrow A' \) between two lexicographically ordered binary trees \( \langle A,\prec,\preceq \rangle \) and \( \langle A',\prec',\preceq' \rangle \) is said to be an \emph{s-hierarchical mapping} if for all \( x \in A \), \( f(x) = \{ f(L)|f(R) \}^A \) whenever \( x = \{ L|R \}^A \). If, in addition, \( A \) and \( A' \) are s-hierarchical ordered fields and the mapping is also an embedding of ordered fields, it is said to be an \emph{s-hierarchical embedding} or an s-hierarchical isomorphism.

It is not difficult to show that a mapping \( f:A \rightarrow A' \) between two lexicographically ordered binary trees \( \langle A,\prec,\preceq \rangle \) and \( \langle A',\prec',\preceq' \rangle \) is s-hierarchical if and only if the mapping preserves order, the predecessor relation and tree-rank [Ehrlich 2001, Lemma 1, p. 1238].\(^{14}\) This in conjunction with the structure of s-hierarchical ordered fields underlies the following result that brings to the fore just how distinguished s-hierarchical mappings of s-hierarchical ordered fields really are.

\(^{14}\) That is, for all \( x,y \in A \): (i) \( f(x) \prec f(y) \) whenever \( x \prec y \); (ii) \( f(x) \prec'_A f(y) \) whenever \( x \prec_A y \); and (iii) \( \rho_A(x) = \rho_A(f(x)) \).
Theorem 6 [Ehrlich 2001, Lemma 2]. Every s-hierarchical mapping between s-hierarchical ordered fields is an s-hierarchical embedding; moreover, if \( f: A \to S \) and \( g: A \to S \) are s-hierarchical mappings, then \( f = g \) and \( f(A) \) is an initial subtree of \( S \).

In virtue of Theorem 6, a mapping \( f: A \to S \) between s-hierarchical ordered fields is an s-hierarchical embedding if and only if it is an s-hierarchical mapping; it is also the unique such mapping; moreover, its image is an ordered field that is an initial subtree of \( S \). Henceforth, we will refer to such an ordered field as an initial subfield. An initial subfield of an s-hierarchical ordered field is, in fact, an s-hierarchical ordered field; furthermore, an ordered field admits a relational extension to an s-hierarchical ordered field if and only if it is isomorphic as an ordered field to an initial subfield of an s-hierarchical ordered field [Ehrlich 2001].

In the following section the closely related notions of s-hierarchical embedding and initial subfield play important roles in shedding further light on \( No \)'s structure as an s-hierarchical ordered field.

7. CHARACTERIZATIONS OF THE S-HIERARCHICAL ORDERED FIELD OF SURREAL NUMBERS

In addition to being (up to isomorphism) the unique Dedekind complete ordered field, the ordered field of real numbers is (up to isomorphism) both the unique universal and the unique maximal (or non-extensible), Archimedean ordered field, the condition of maximality or non-extensibility being Hilbert’s continuity condition. These theorems provide alternative characterizations of the ordered field of reals (up to isomorphism) as the unique ordered field 

**having all possible gradations consistent with the Archimedean axiom.** Theorem 7 below provides analogs of these three classical results for \( No \) considered as an s-hierarchical ordered field.
Mimicking the classical conceptions, an $s$-hierarchical ordered field $A$ will be said to be *universal* if for each $s$-hierarchical ordered field $B$ there is an $s$-hierarchical embedding $f: B \to A$, and it will be said to be *maximal* (or *non-extensible*) if there is no $s$-hierarchical ordered field $B$ that properly contains $A$ as an initial subfield. In addition, an $s$-hierarchical ordered field $A$ will be said to be *complete* if it is complete as a lexicographically ordered binary tree, i.e., if $\{L|R\}$ exists whenever $L$ and $R$ are subsets of $A$ for which $L < R$.

**Theorem 7** [Ehrlich 2001]. Let $A$ be an $s$-hierarchical ordered field. $A$ is complete if and only if $A$ is universal if and only if $A$ is maximal if and only if $A$ is isomorphic to $\mathbb{N}_o$.

8. THE $S$-HIERARCHICAL ORDERED FIELD OF REAL NUMBERS

Thus far, the only $s$-hierarchical ordered fields whose existence we have discussed are complete $s$-hierarchical ordered fields. However, there is a vast array of $s$-hierarchical ordered fields that fall short of completeness, the full spectrum of which was characterized by the author in [Ehrlich 2001, § 4]. It is a consequence of Theorems 6 and 7 together with some related observations from Section 6 that revealing the spectrum of $s$-hierarchical ordered fields reduces to revealing the spectrum of initial subfields of $\mathbb{N}_o$. While we will make no attempt in this paper to spell out the entire spectrum, in this and the following section we will draw attention to components of the spectrum that are of particular importance from the standpoints of the classical, elementary and absolute theories of continua. In this section, we introduce the $s$-hierarchical ordered field of real numbers, a structure that, as we shall later see, plays a critical role in assigning canonical appellations to arbitrary surreal numbers.
**Definition 4.** Let $\mathbb{D}$ be the set of all surreal numbers having finite tree-rank and further let $\mathbb{R} = \mathbb{D} \cup \{X|Y|(X,Y) \text{ is a Dedekind gap in } \mathbb{D} \}.^{15}$

Except for inessential changes, the following result is due to Conway [1976, pp. 23-25].

**Proposition 4.** $\mathbb{R}$ (with $+,-,\cdot$ and $<$ defined à la No) is isomorphic to the ordered field of reals defined in any of the more familiar ways, $\mathbb{D}$ being No’s ring of dyadic rationals (i.e., rationals of the form $m/2^n$ where $m$ and $n$ are integers); $n = \{0,\ldots, n-1\}$ for each positive integer $n$, $-n = \{-(n-1),\ldots,0\}$ for each positive integer $n$, $0 = \{\}$, and the remainder of the dyadics are the arithmetic means of their left and right predecessors of greatest tree-rank; e.g., $\mathbb{N}_2 = \{011\}$.

In virtue of Proposition 4, the first few levels of the surreal number tree may be depicted as follows where, as in Figure 2, the strictly descending sequences of dots connect a member of the tree to its predecessors.

![Figure 4](image)

Being an ordered field, No contains exactly one subring of integers, one subring of dyadic rationals, and one subfield of rationals. On the other hand, as we alluded to

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$^{15}$ A Dedekind gap $(X,Y)$ in $\mathbb{D}$ is a partition of $\mathbb{D}$ into two nonempty subsets $X$ and $Y$ such that $X < Y$, $X$ has no greatest member, and $Y$ has no least member.
above, \(\mathbb{N}o\) contains an entire proper class of distinct isomorphic copies of the ordered field of reals. What distinguishes \(\mathbb{R}\) from the rest is captured by the first part of

**Theorem 8** [Ehrlich 2001]. \(\mathbb{R}\) is the unique isomorphic copy of the ordered field of reals that is an initial subfield of \(\mathbb{N}o\). In fact, every Archimedean ordered field is isomorphic to exactly one initial subfield of \(\mathbb{N}o\), the latter being an initial subfield of \(\mathbb{R}\).

The first part of Theorem 8 being its requisite justification, henceforth we shall refer to \(\mathbb{R}\) as the set of real numbers. In accordance with this convention, \(\mathbb{R}\) is the set of all surreal numbers whose sign-expansions have length \(\leq \omega\) that end neither in an infinite sequence of \(-s\) nor in an infinite sequence of \(+s\). The real numbers whose sign-expansions have length \(<\omega\) are the dyadic rationals, and the remaining rational numbers are the members of \(\mathbb{R}\) having sign-expansions that end with finite sequences of signs (containing at least one \(+\) and at least one \(-\)) that recur indefinitely. For instance, \(\frac{1}{3}\) is the surreal number having the sign-expansion \(+\underline{\ldots}+\) where the underlining indicates the finite sequence of signs that indefinitely recurs. For a simple rule, due to Elwyn Berlekamp, for obtaining the value of a real number from its sign-expansion, see [Conway 1976, p. 31; 2001, p. 31].

9. S-HIERARCHICAL ARCHIMEDEAN AND ABSOLUTE ARITHMETIC CONTINUA

As was noted above, \(\langle \mathbb{R}, +, ; , <, 0, 1 \rangle\) is (up to isomorphism) both the unique universal and the unique non-extensible Archimedean ordered field. Analogs of these results for ordered fields, however, fail to hold for \(\langle \mathbb{N}o, +, ; , <, 0, 1 \rangle\) [Ehrlich 2001: Proposition 3]. On the other hand, as Theorems 6-8 show, analogous results do in fact hold for the s-hierarchical counterparts of \(\langle \mathbb{R}, +, ; , <, 0, 1 \rangle\) and \(\langle \mathbb{N}o, +, ; , <, 0, 1 \rangle\). This being the case, the latter results may be encapsulated as follows, where a *universal Archimedean s-hierarchical ordered field* and a *maximal Archimedean s-hierarchical
ordered field are defined in the expected manner as straightforward Archimedean variants of universal and maximal s-hierarchical ordered fields.

**Theorem 9.** (I) The following sets of axioms constitute (categorical) axiomatizations of $\langle \mathbb{R},+,\cdot,\prec,\preceq,0,1 \rangle$; (II) by deleting the Archimedean axiom from the following axiomatizations one obtains categorical axiomatizations of $\langle \mathbb{N}_0,+,\cdot,\prec,\preceq,0,1 \rangle$.

Axioms for s-hierarchical ordered fields

Archimedean axiom

Axiom of Arithmetric Universality

or

Axiom of Arithmetric Maximality

s - Hierarchical Continuity Axioms

whereby the Axiom of Universality (Maximality) we mean the assertion: The collection of numbers together with the corresponding relations defined on it constitutes a universal (maximal) model of the above stated axioms.

10. S-HIERARCHICAL ELEMENTARY ARITHMETIC CONTINUA

Unlike the Archimedean ordered fields, not every ordered field is isomorphic to an initial subfield of $\mathbb{N}_0$. On the other hand, as we mentioned in the Introduction, and as we shall now see, every ordered field has a theoretically significant extension to an ordered field that is.

Following Emil Artin and Otto Schreier [1926/1965], an ordered field $A$ may be said to be *real-closed* if it admits no extension to a more inclusive ordered field that results from supplementing $A$ with solutions to polynomial equations with coefficients in
Intuitively speaking, real-closed ordered fields are precisely those ordered fields having no holes that can be filled by algebraic means alone. The ordered field of real numbers is the paradigmatic real-closed ordered field; indeed, it was the desire to capture the algebraic theory of the reals that motivated the development of Artin and Schreier’s theory in the first place [Sinaceur 1994; 2006]. Tarski [1939/1986, 1948/1986] later shed important model-theoretic light on Artin and Schreier’s algebraic conception by showing that real-closed ordered fields are precisely the ordered fields that are first-order indistinguishable from the ordered field of reals. For this reason they are sometimes called *elementary continua*.

Unlike the classical arithmetic continuum, the ordered field of rational numbers is not real-closed--it can be extended to a richer ordered field by supplementing the rationals with solutions to polynomial equations with rational coefficients. The richest ordered field that can be obtained from the rationals in this fashion is to within isomorphism the ordered field of real algebraic numbers. The system of real algebraic numbers is in fact (up to isomorphism) the smallest real-closed ordered field containing the ordered field of rationals as a subfield. Among the important discoveries to emerge from the theory of real-closed ordered fields is that the relation the rationals bear to the real algebraic numbers is a special case of a far more general relation. In particular, by a celebrated result of Artin and Schreier [1926/1965], every ordered field $A$ has (up to isomorphism) a unique *real-closure*, i.e. a smallest real-closed ordered field containing $A$. If $A$ is itself real-closed, then $A$ is its own real-closure, otherwise the real-closure of $A$ is a (cardinality preserving) proper extension of $A$.

When viewed against the backdrop of the theory of elementary continua, the following result goes some distance toward showing that the spectrum of initial subfields of $\mathbb{N}$ is as theoretically revealing as it is robust.

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16 For a brief overview of some of the seminal results in the theory of real-closed ordered field, see [Chang and Keisler 1990, pp. 342-348]; and for the historical development and philosophical significance of these important structures, see [Sinaceur 1994 and 2006].
Theorem 10 [Ehrlich 2001]. Every real-closed ordered field is isomorphic to an initial subfield of $\mathbf{No}$. In fact, $\mathbf{No}$ (which was shown by Conway [1976] to be real-closed) is the union of a chain of increasingly more inclusive real-closed ordered surreal number fields each of which is a proper initial subfield of $\mathbf{No}$ whose universe is a set. Indeed, every real-closed initial subfield of $\mathbf{No}$ whose universe is a set is a component of such a chain.

11. EVERY SURREAL NUMBER HAS ITS OWN PROPER NAME

As we mentioned in the Introduction, each surreal number can be assigned a canonical proper name that is a reflection of its characteristic individual properties. In order to define these Conway names, as we call them, we require a number of definitions beginning with the following classical ones applied to $\mathbf{No}$.17

Two elements $a$ and $b$ of $\mathbf{No}$ are said to be Archimedean equivalent, written $a \approx b$, if there are positive integers $m$ and $n$ such that $m|a| > |b|$ and $n|b| > |a|$; if $a \neq b$ and $|a| < |b|$, then we write $|a| < < |b|$ and $a$ is said to be infinitesimal (in absolute value) relative to $b$ and $b$ is said to be infinite (in absolute value) relative to $a$; the class of all members of $\mathbf{No}$ that are Archimedean equivalent to some member of $\mathbf{No}$ is said to constitute an Archimedean class of $\mathbf{No}$. 0, which is infinitesimal (in absolute value)

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17 In this section of the paper there is a presupposition that every real number can be assigned its own proper name. There are several simple ways in which this can be done including letting the sign-expansion of a real number serve as its proper name. Such an assignment of proper names could, of course, be extended to the surreal numbers more generally. However, as will soon be evident, the assignment of “Conway names” is a good deal more revealing.

What we call the “Conway name” of a surreal number, Conway calls the “normal form”. The introduction of Conway names does not make full use of $\mathbf{No}$’s structure as an s-hierarchical ordered field, but rather its structure as an s-hierarchical ordered vector space over the s-hierarchical ordered field of real numbers [Ehrlich 2001, pp. 1236, 1241-1243].
relative to every other surreal number, is the sole surreal number that is not a member of an Archimedean class.

Following Conway, an element of $No$ is said to be a leader if it is the simplest member of the positive elements of an Archimedean class of $No$. Since the class of positive elements of an Archimedean class of $No$ is a convex subclass of $No$, the concept of a leader is well defined. Henceforth, by $Lead(No)$ we mean the class of all leaders of $No$.

One of the salient features of $Lead(No)$ is given by

**Theorem 11** [Conway 1976; Ehrlich 2001]. $\langle Lead(No),<_{No},\mid Lead(No),<,\mid Lead(No)\rangle$ is a lexicographically ordered full binary tree and, as such, the unique s-hierarchical mapping from $Lead(No)$ to $No$ (henceforth, $\Phi_{No}$) is a bijection.

The following recursive definition--which is a variation on an idea due to Conway [1972, p. 31]--provides a unique appellation for each leader in $No$. In effect, it assigns the appellation “$\omega^0$” to $1$, which is the simplest leader in $No$, the appellation “$\omega^{-1}$” to the simplest leader in $No$ less than $\omega^0$, the appellation “$\omega^1$” to the simplest leader in $No$ greater than $\omega^0$, and so on.

**Definition 5.** For each $y \in No$, $\omega^y = \{0, no^{L_{S(y)}} | \frac{1}{2^n} o^{R_{S(y)}} \}^{No}$, where $n$ is understood to range over all positive integers, and $\omega^{L_{S(y)}}$ and $\omega^{R_{S(y)}}$ are understood to denote typical members of $\{\omega^x : x \in L_{S(y)}\}$ and $\{\omega^x : x \in R_{S(y)}\}$, respectively.

Justification for the contention that Definition 5 provides a unique appellation to each leader in $No$ accrues from

**Theorem 12** [Ehrlich 2001]. $\omega^y = \Phi_{No}^{-1}(y)$, for each $y \in No$. 

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Using a classical argument, one may show that if \( x \in No - \{0\} \), there is a unique \( r \in \mathbb{R} - \{0\} \), a unique \( y \in No \), and a unique \( a \in No \) such that \( x = r \omega^y + a \) where \( a \) is infinitesimal (in absolute value) relative to \( \omega^y \). To put this somewhat differently, if \( x \in No - \{0\} \), then there is a unique \( r \in \mathbb{R} - \{0\} \) and a unique \( y \in No \) for which \( |x - r \omega^y| << \omega^y \). This elementary observation plays pivotal roles in both the formulation and proof of the following s-hierarchical recasting of an argument sketched by Conway.

**Theorem 13** [Ehrlich 2001]. For each \( x \in No - \{0\} \), there is a unique descending chain \( I_x^\alpha, \alpha < \beta \in On \), of convex subclasses of \( No \) whose intersection contains \( x \) as its simplest member, and whose components are defined by recursion as follows: \( I_x^\alpha \) is the subclass of all members of \( No \) of the form \( s_a + r_a \omega^y + a_a \) where

(i) \( \begin{cases} s_a = 0, & \text{if } \alpha = 0 \\ s_a \text{ is the simplest member of } \bigcap_{\gamma < \alpha} I_x^\gamma, & \text{otherwise} \end{cases} \)

(ii) \( r_a \omega^y \) --henceforth called the \( \alpha \)-term of \( x \)-- is the unique member of \( No \) for which \( r_a \in \mathbb{R} - \{0\}, y_a \in No \) and \( |x - (s_a + r_a \omega^y)| << \omega^y \);

(iii) \( |a_a| << \omega^y \).

Since the intersection of a chain of convex subclasses is itself a convex subclass, it follows from Theorem 13, that for each \( x \in No - \{0\} \) there is a unique convex subclass of \( No \), containing \( x \) as its simplest member, consisting of all and only those surreal numbers whose \( \alpha \)-term is \( r_a \omega^y \) for all \( \alpha < \) some ordinal \( \beta \) that depends on \( x \). With this as its underlying justification, the completion of the assignment of appellations to surreal numbers begun with Definition 5 is carried out via

**Definition 6.** We will refer to the formal expression

\[
\sum_{\alpha < \beta} \omega^y \cdot r_a
\]

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as the *Conway name* of a surreal number $x$, treat the Conway name of $x$ as a proper name of $x$ and, accordingly, write \( x = \sum_{\alpha < \beta} \omega^{y_{\alpha}} \cdot r_{\alpha} \) if and only if either $x = 0$ and $\beta = 0$, or $x$ is the simplest surreal number whose $\alpha$-term is $r_{\alpha}\omega^{y_{\alpha}}$ for all $\alpha < \beta$. (Moreover, we will delete the summation sign or insert sum signs between successive terms when it is convenient to do so, as well as write $\omega^0 \cdot r = r$, $\omega^y \cdot -r = -\omega^y \cdot r$, $\omega^1 \cdot 1 = \omega$, $\omega^{1/2} = \frac{\omega}{2}$, $\omega^{-y} = \frac{1}{\omega^y}$ and so on, in accordance with standard practice).

As is well known, in virtue of a celebrated theorem due to Cantor and Hessenberg, every ordinal has a unique *Cantor normal form*, i.e., a unique representation of the form

\[
\sum_{\alpha < n} \omega^{y_{\alpha}} \cdot a_{\alpha}
\]

where \( \{y_{\alpha}; \alpha < n\} \) is a (possibly empty) finite descending sequence of ordinals, \( \{a_{\alpha}; \alpha < n\} \) is a sequence of finite ordinals $> 0$, and the ‘$c$’ affixed to the various operations indicates that the operations are the familiar Cantorian operations on ordinals. The following result shows, among other things, that Conway names of surreal numbers generalize the Cantor normal forms of ordinals.

**Theorem 14** [Conway 1972; Ehrlich 2001]. Every surreal number has a Conway name; distinct surreal numbers have distinct Conway names; furthermore, the formal expression

\[
\sum_{\alpha < \beta} \omega^{y_{\alpha}} \cdot r_{\alpha}
\]

is the Conway name of some surreal number if and only if \( \{y_{\alpha}; \alpha < \beta \in On\} \) is a (possibly empty) descending sequence of members of $No$ and \( \{r_{\alpha}; \alpha < \beta\} \) is a sequence of nonzero real numbers. In addition, the Conway name of an ordinal is just its Cantor normal form.
Conway names provide highly perspicuous representations of surreal numbers. \( \omega - \pi \) is indeed the surreal number obtained by subtracting \( \pi \) from \( \omega \), \( \frac{\omega}{2} \) is \( \omega \) divided by 2, \( \frac{1}{\omega} \) is the multiplicative inverse of \( \omega \), \( \frac{1}{\sqrt{2}} \) is the cube root of \( \frac{1}{\omega^3} \), \( \omega^{\omega} + \omega.2+1 \) and \( \sqrt{2} \) are the ordinal and real number respectively so named, and so on. Even Conway names indexed over all \( \alpha < \) some infinite limit ordinal \( \beta \) encode significant algebraic-tree theoretic information about the surreal numbers they denote but we will not develop this here.

Besides providing perspicuous representations, Conway names make surreal numbers more tractable from an algebraic point of view. Indeed, when surreal numbers are denoted by their respective Conway names, the basic field operations can be performed on them analogously to the familiar operations on polynomials as can the familiar ordering of elements by first-differences, the identification of multiplicative inverses and the extraction of \( n \)-th roots. In the case of addition and multiplication, in particular, the operations are the familiar termwise operations where terms having a common exponent are added by the rule

\[
\omega^x \cdot a +_H \omega^x \cdot b = \omega^x \cdot (a + b)
\]

where \( a + b \) is the ordinary sum of real numbers, and the product of an arbitrary pair of terms is given by

\[
\omega^x \cdot a \cdot_1 \omega^y \cdot b = \omega^{x+y} \cdot ab
\]

where \( ab \) is the ordinary product of real numbers and \( x +_1 y \) is the aforementioned termwise sum. So, for example, given two surreal numbers \( \omega - \frac{1}{2} \) and \( 1 + \frac{1}{\omega} \), their sum is given by \( \omega + \frac{1}{2} + \frac{1}{\omega} \) and their product by \( \omega - \frac{1}{2} + \frac{1}{\omega^2} \). Moreover, \( 1 + \frac{1}{\omega} < \omega - \frac{1}{2} \) since at the first corresponding term in which \( \omega^1.0 + \omega^0.1 + \omega^{-1}.1 \) and \( \omega^1.1 + \omega^0.\frac{1}{2} + \omega^{-1}.0 \) differ (i.e., \( \omega^1.0 \) and \( \omega^1.1 \)) \( 0 < 1 \).
Underwriting the preceding observations regarding the representational and algebraic virtues of Conway names is the following result where the operations \(+_H\) and \(\cdot_H\) are defined on, and the relation \(<_H\) is defined between, surreal numbers denoted by their respective Conway names (supplemented with “dummy” terms with zeros for coefficients to permit a uniform representation of all surreal numbers).

**Theorem 15** [Conway 1976; Ehrlich 2001]. \(\langle \text{No}, +_H, \cdot_H, <_H, , 0, 1 \rangle = \langle \text{No}, +, \cdot, <, , 0, 1 \rangle\), when \(+_H\), \(\cdot_H\) and \(<_H\) are defined as follows:

\[
\sum_{y \in \text{No}} \omega^y \cdot a_y +_H \sum_{y \in \text{No}} \omega^y \cdot b_y = \sum_{y \in \text{No}} \omega^y \cdot (a_y + b_y),
\]

\[
\sum_{y \in \text{No}} \omega^y \cdot a_y \cdot_H \sum_{y \in \text{No}} \omega^y \cdot b_y = \sum_{y \in \text{No}} \omega^y \cdot \left[ \sum_{(\mu, \nu) \in \text{No} \times \text{No}} a_\mu b_\nu \right],
\]

\[
\sum_{y \in \text{No}} \omega^y \cdot a_y <_H \sum_{y \in \text{No}} \omega^y \cdot b_y, \text{ if } a_y = b_y \text{ for all } y \text{ some } x \in \text{No} \text{ and } a_x < b_x.
\]

When restricted to ordinals, \(+_H\) and \(\cdot_H\) are not of course the familiar non-commutative sums and products of Cantor, but rather the so-called *natural sums* and *natural products* of ordinals due to Hessenberg and Hausdorff, respectively, and \(<_H\) is the standard ordering of ordinals (written in Cantor normal form) by first-differences.\(^{18}\) This might lead the reader suspect that the subscript “\(H\)” affixed to symbols denoting the field operations and ordering relation are a tribute to Hessenberg and Hausdorff. However, while no slight is intended, this is not the case--after all, Cantor normal forms of ordinals are always indexed over finite ordinals. Rather, the subscripts are in honor of Hans Hahn [1907], whose corresponding field operations and ordering relation defined on transfinite power series are the *loci classici* for the types of operations and ordering relation defined in Theorem 15. In fact, Theorem 15 essentially shows that \(\text{No}, \)

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\(^{18}\) For references on, and remarks regarding the historical background of, natural sums and products of ordinals, see [Ehrlich 2006, pp. 24-25 (including Note 40)].
considered as an ordered field, is isomorphic to a particular ordered field of formal power series with coefficients in \( \mathbb{R} \) and exponents in \( \text{No} \) (considered as an ordered additive group) that can be obtained via a classic construction of Hahn [Ehrlich 1988; 2001].

12. THE ORDERED FIELD OF SURREAL NUMBERS

Since there is a multitude of real-closed ordered fields, it is natural to inquire if, like \( \mathbb{R} \), it is possible to distinguish \( \text{No} \) (to within isomorphism) from the remaining real-closed ordered fields by appealing solely to its order. As we shall now see, the following definition enables one to do just this.

Definition 7. An ordered class \( \langle A, \prec \rangle \) will be said to be an absolute linear continuum if for all sub-sets \( L \) and \( R \) of \( A \) where \( L < R \) there is a \( y \in A \) such that \( L < \{y\} < R \).

An absolute linear continuum \( \langle A, \prec \rangle \) is both absolutely dense in the sense that for each pair of nonempty sub-sets \( L \) and \( R \) of \( A \) where \( L < R \), there is a \( y \in A \) such that \( L < \{y\} < R \), and absolutely extensive in the sense that given any subset \( L \) of \( A \) there are members \( a \) and \( b \) of \( A \) that are respectively smaller than and greater than every member of \( L \). In fact, since in the definition of an absolute linear continuum \( L \) or \( R \) may be empty, one can readily show that an ordered class is an absolute linear continuum if and

\[19\] In [Ehrlich 1988; 1992; 1994] and a number of other works, we refer to these structures \( \eta_{\text{on}} \)-orderings because they extend to proper classes Hausdorff’s [1909; 1914] idea of an \( \eta_{\prec} \)-ordering of power \( \mathcal{N}_{\prec} \), that is an ordered set \( A \) of power \( \mathcal{N}_{\prec} \) such that for all sub-sets \( L \) and \( R \) of \( A \) where \( L < R \) and \( |L|, |R| \leq \mathcal{N}_{\prec} \) there is a member of \( A \) lying strictly between those of \( L \) and those of \( R \).

\[20\] The reader will notice that a lexicographically ordered binary tree is an absolute linear continuum if and only if it is complete (in the sense of Section 4). Accordingly, just as the s-hierarchical ordered field of real numbers is distinguished (up to isomorphism) by the ordering of its elements, the s-hierarchical ordered field of surreal numbers is distinguished (up to isomorphism) by the ordering of its elements.
only if it has both of the just-stated properties.\textsuperscript{21} Thus, since every element of an ordered class must either lie between two of its nonempty subclasses or lie to the right or to the left of some (possibly empty) subclass, absolute linear continua are ordered classes having \textit{no order-theoretic limitations that are definable in terms of sets of standard set theory}. 

In 1895 Cantor provided an order-theoretic characterization of the so-called \textit{linear continuum} and showed that $\mathbb{R}$ considered as an ordered class is (up to isomorphism) the unique such structure. The following is the analog for absolute linear continua of Cantor’s result.

\textbf{Theorem 16.} [Ehrlich 1988: Lemma 1]. $\langle \text{No},< \rangle$ is (up to isomorphism) the unique absolute linear continuum.

Unlike the ordered field of reals, however, the ordered field of surreal numbers is not distinguished (up to isomorphism) from other ordered fields by its structure as an ordered class. Indeed, there are infinitely many pairwise nonisomorphic ordered fields that are absolute linear continua.\textsuperscript{22} Happily, however, what one can prove is

\textbf{Theorem 17.} [Ehrlich 1988: Lemma 2]. $\langle \text{No},+,\cdot,<,0,1 \rangle$ is (up to isomorphism) the unique real-closed ordered field that is an absolute linear continuum.

In virtue of Theorem 17, No is not only devoid of set-theoretically defined order-theoretic limitations, it is devoid of algebraic limitations as well; moreover, to within

\textsuperscript{21} In a number of earlier works [Ehrlich 1987, p. 243; 1992, p. 172; 1994a, p. xxvii], we described the property employed in the definition of an absolute linear continuum as a condition of absolute density. While we continue to believe this is justifiable, we now believe that at least for some purposes it is more revealing to split the condition into absolute density and absolute extensivity, as we have done here.

\textsuperscript{22} Since every $\alpha$-saturated ordered field is an $\eta_{\alpha}$-ordering, to obtain such an ordered field one need only form the union of a continuous chain $A_\alpha$ ($\alpha < \text{On}$) of ordered fields where $A_0$ is not real-closed and $A_{\alpha+1}$ is an $\alpha+1$-saturated elementary extension of $A_\alpha$ for each $\alpha$. See, for example, [Chang and Keisler 1990, ch. 5].
isomorphism, it is the unique ordered field that is devoid of both types of limitations or ‘holes’, as they might more colloquially be called. That is, \( \mathbb{N} \) not only exhibits all possible algebraic and set-theoretically defined order-theoretic gradations consistent with its structure as an ordered field, it is to within isomorphism the unique such structure that does. It is ultimately this together with a number of closely related results that underlies our contention that \( \mathbb{N} \) may be naturally regarded as an absolute arithmetic continuum (modulo NBG).

13. THE ORDERED FIELDS OF REAL AND SURREAL NUMBERS

Before concluding our discussion of \( \langle \mathbb{N},+,\cdot,\times,0,1 \rangle \), however, we wish to shed a bit more light on its inclusiveness as an ordered number field, its relation to the classical arithmetic continuum, and its amorphousness as a system of numbers great and small.

To begin with, every categorical characterization of the ordered field of surreal numbers expresses in one way or another just how rich in types of numbers great and small \( \mathbb{N} \) really is. Unlike the one offered in Theorem 17, the two provided in Theorem 18 below are of a model-theoretic nature. The first is based upon

**Definition 8.** An ordered field \( A \) is said to be universally extending if for each ordered subfield \( B \) of \( A \) of power \(<\mathcal{O}n\) and each ordered field \( A' \) extending \( B \), there is an isomorphism \( f:A' \to A \) that is an extension of the identity map on \( B \).

Intuitively speaking, \( A \) is a universally extending ordered field if every possible way of enriching the ordered field-theoretic gradations of an ordered subfield \( B \) of \( A \) that is consistent with NBG is already (isomorphically) realized as an extension of \( B \) in \( A \), if the universe of \( B \) is a set. The reader will notice that insofar as every ordered field is an
extension of the ordered field of rationals, every universally extending ordered field contains an isomorphic copy of every ordered field. There is, however, a plethora of ordered fields that are inclusive in this sense but are not universally extending. However, as we shall shortly see, the following definition provides the means to distinguish between the universally extending ordered fields and those that are “merely” universal.

**Definition 9.** An ordered field $A$ is said to be *homogeneous-universal* if it is *universal*—every ordered field can be embedded in $A$, and it is *homogeneous*—every isomorphism between ordered subfields $B$ and $B'$ of $A$ of power $< \aleph_0$ can be extended to an automorphism of $A$.

Since an automorphism of $A$ is an isomorphism of $A$ onto itself, *the homogeneity condition essentially ensures that any pair of ordered subfields of $A$ whose universes are sets that are structurally indistinguishable have structurally indistinguishable surroundings as well. As a result, any role that can be played in $A$ by a given ordered
subfield whose universe is a set can be played equally well by any of its other isomorphic counterparts in $A$.

Mimicking the corresponding definitions for ordered fields, an Archimedean ordered field $A$ is said to be *homogeneous-universal* if every Archimedean ordered field can be embedded in $A$ and every isomorphism between ordered subfields of $A$ can be extended to an automorphism of $A$; and an Archimedean ordered field $A$ is said to be *universally extending* if for each ordered subfield $B$ of $A$ and each Archimedean ordered field $A'$ extending $B$, there is an isomorphism $f : A' \to A$ that is an extension of the identity map on $B$.\(^{23}\)

The following theorem helps to bring to the fore the intimate relation between the classical arithmetic continuum and the purported absolute arithmetic continuum considered as inclusive ordered number fields.

**Theorem 18** [Ehrlich 1987; 1989a; 1992]. (I) The following sets of axioms constitute (categorical) axiomatizations of $\langle \mathbb{R}, +, \cdot, <, 0, 1 \rangle$; (II) by deleting the Archimedean axiom from the following axiomatizations one obtains categorical axiomatizations of $\langle \mathbb{N}_0, +, \cdot, <, 0, 1 \rangle$.

Axioms for ordered fields

Archimedean axiom

Axiom of (Arithmetic) Homogeneous - Universality

or, alternatively

Axiom of (Arithmetic) Universal Extensibility

Continuity Axioms

whereby the Axiom of Arithmetic Homogeneous-Universality (Axiom of Arithmetic Universal Extensibility) we mean the assertion: The collection of numbers together with

---

\(^{23}\) The reader will note that in the above two definitions implicit use is made of the fact that every subfield of an Archimedean ordered field is itself an Archimedean ordered field whose universe is a set.
the corresponding relations defined on it constitutes a Homogeneous-Universal (Universally Extending) model of the above stated axioms.

Intuitively speaking, Theorem 18 asserts that whereas \( \langle \mathbb{R}, +, <, 0, 1 \rangle \) is (up to isomorphism) the unique ordered number field containing all possible types of numbers great and small modulo the Archimedean axiom, \( \langle \mathcal{N}_0, +, <, 0, 1 \rangle \) is (up to isomorphism) the unique ordered number field containing all possible types of numbers great and small modulo NBG.\(^{24}\) We believe we are justified in referring to the pair of alternative axioms that underlie this interpretation as continuity axioms since in the context of the above axiomatizations they are equivalent to any of the more familiar continuity conditions including those due to Cantor, Dedekind and Hilbert.

We hasten to emphasize, however, that the import of the homogeneity portion of the first continuity condition differs from the Archimedean case to the non-Archimedean case. This is a consequence of the fact that whereas \( \langle \mathbb{R}, +, <, 0, 1 \rangle \) contains exactly one isomorphic copy of each Archimedean ordered field, \( \langle \mathcal{N}_0, +, <, 0, 1 \rangle \) contains a plethora of isomorphic copies of ordered fields whose universes are sets.\(^{25}\) Thus, whereas the homogeneity condition holds vacuously in \( \langle \mathbb{R}, +, <, 0, 1 \rangle \), it points to genuine amorphousness in \( \langle \mathcal{N}_0, +, <, 0, 1 \rangle \), an amorphousness that is not present in \( \langle \mathcal{N}_0, +, <, 0, 1 \rangle \).

14. ABSOLUTELY CONTINUOUS ELEMENTARY EUCLIDEAN GEOMETRY

\(^{24}\) Another important way of lending precision to the idea that \( \langle \mathcal{N}_0, +, <, 0, 1 \rangle \) is (up to isomorphism) the unique ordered number field containing all possible types of numbers great and small modulo NBG is to say that \( \langle \mathcal{N}_0, +, <, 0, 1 \rangle \) is (up to isomorphism) the unique absolutely saturated model for the theory of real-closed ordered fields [Ehrlich 1989].

\(^{25}\) In fact, \( \langle \mathcal{N}_0, +, <, 0, 1 \rangle \) also properly contains a plethora of isomorphic copies of ordered fields whose universes are proper classes, including a plethora of isomorphic copies of \( \langle \mathcal{N}_0, +, <, 0, 1 \rangle \) itself! The latter observation is a simple consequence of [Ehrlich 2001: Proposition 3].
The relation between \( \langle \text{No}, +, \cdot, <, 0, 1 \rangle \) and Euclidean geometry can be fleshed out using any of the standard Euclidean geometrical frameworks. For our purpose here, however, Tarski’s system \( P \) [Tarski 1959; Schwabhäuser, Szmielew, and Tarski 1983] provides an especially convenient starting point.

In Tarski’s system \( P \) only points are treated as individuals and the only predicates employed in the axioms are a ternary predicate \( B \) (where “\( Bxyz \)” is read \( y \) lies between \( x \) and \( z \) [the case when \( y \) coincides with \( x \) or \( z \) not being excluded]) and a quarternary predicate \( \equiv \) (where “\( xy \equiv zu \)” is read \( x \) is as distant from \( y \) as \( z \) is from \( u \)). Over the years, Tarski’s system of axioms for \( P \) has undergone substantial evolution, culminating in a set of ten axioms (cf. [Schwabhäuser, Szmielew, and Tarski 1983; Ehrlich 1997, p. 61]) that are collectively equivalent to Hilbert’s classical axioms of order, incidence, congruence and parallelism [1899; 1971]. That is, Tarski’s system of axioms for \( P \), henceforth \( A_1 - A_{10} \), is equivalent to Hilbert’s system of axioms for Euclidean geometry save Hilbert’s continuity axioms—the axioms that limit the models of Hilbert’s other axioms to isomorphic copies of classical Cartesian geometry (over the reals). In place of the latter axioms, Tarski supplements \( A_1 - A_{10} \) with an \textit{elementary continuity scheme}, henceforth \( A_{11} \), that restricts the models of \( A_1 - A_{10} \) to those that are elementary equivalent to classical Cartesian geometry. The resulting system of geometry based on \( A_1 - A_{11} \), henceforth \( E \), is referred to as \textit{elementary Euclidean geometry}.

The conception that bridges the gap between the domains of number and Euclidean magnitude is the classical notion of a Cartesian space over an ordered field. In Tarski’s framework, this familiar concept assumes the following simple form.

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A Cartesian space \( C_2(F) \) over an ordered field \( \langle F, +, \cdot, \leq \rangle \) is a structure \( \langle A_F, B_F, \equiv_F \rangle \) where the relations \( B_F \) and \( \equiv_F \) are defined on \( A_F = \{ (x_1, x_2) : x_1, x_2 \in F \} \) by the stipulations:

\[
\begin{align*}
B_F \langle x_1, x_2, x_3 \rangle & \iff \text{there is a } \lambda \in F \text{ for which } 0 \leq \lambda \leq 1 \text{ and } y_\alpha - x_\alpha = \lambda(z_\alpha - x_\alpha) \text{ for } \alpha = 1, 2; \\
\equiv_F \langle x_1, x_2 \rangle & \iff \sum_{\alpha=1}^2 (x_\alpha - y_\alpha)^2 = \sum_{\alpha=1}^2 (u_\alpha - v_\alpha)^2.
\end{align*}
\]

The relation between Cartesian spaces and models of \( E \) is given by the following celebrated theorem of Tarski: \( M = \langle A, B, \equiv \rangle \) is a model of \( E \) if and only if \( M \) is isomorphic to a Cartesian space over a real-closed ordered field. The “if” part of the proof consists largely of showing that a Cartesian space \( C_2(F) \) over a real-closed ordered field \( F \) is elementary equivalent to \( C_2(\mathbb{R}) \) and is, hence, a model of \( E \). And the proof of the converse essentially consists of two components, the first of which is an appeal to what may be called the fundamental lemma of elementary Cartesian geometry, namely: if \( o \) and \( e \) are distinct points in a model \( M = \langle A, B, \equiv \rangle \) of \( P \) and \( L \) is the line in \( M \) through \( o \) and \( e \), then by letting \( o \) and \( e \) serve as the zero and unit, respectively, and by appealing to the Euclidean theory of proportions which can be derived from \( P \), one can define (in terms of the relations \( B \) and \( \equiv \)) operations \( +_L \) and \( \cdot_L \) on, and a relation \( \leq_L \) between, any two points of \( L \) so that the resulting structure \( F_{oe}^L(M) = \langle L, +_L, \cdot_L, \leq_L, o, e \rangle_M \) is a ordered field for which the following is true\(^{27} \): \( M \) is isomorphic to a Cartesian space over \( F_{oe}^L(M) \). To complete the proof one shows that \( F_{oe}^L(M) \) is real-closed, if Tarski’s elementary continuity scheme holds in \( M \).

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\(^{27}\) The operation \( +_L \) and the relation \( \leq_L \) are defined using restrictions of the relations \( B \) and \( \equiv \) to the line \( L \). To define the multiplicative operation \( \cdot_L \) (which is based on properties of parallel lines) one also appeals to points outside the line. For the definitions of \( +_L, \cdot_L, \text{ and } \leq_L \), see [Tarski 1959, pp. 21-22; and Schwabläuer et. al., 1983, pp. 144-145].
In virtue of the fundamental lemma of elementary Cartesian geometry, the ordered field $F_{ee}^L(M)$, or any isomorphic copy thereof, may be said to be the characteristic field of $M$. Since $\langle \mathbb{N},+,\cdot,\langle,0,1\rangle$ is (up to isomorphism) the unique real-closed ordered field that is an absolute linear continuum, it is evident from the above that if $M$ is a model of $E$, then to ensure that its characteristic field is isomorphic to $\langle \mathbb{N},+,\cdot,\langle,0,1\rangle$ it suffices to ensure that its characteristic field is itself an absolute linear continuum. This may be accomplished by supplementing Tarski’s axioms for elementary geometry with

$A_{12}$. Absolute Continuity Axiom [Ehrlich 1987].

$$\forall X \forall Y [\exists a \forall x \forall y (x \in X \land y \in Y \rightarrow Baxy \land x \neq y) \rightarrow \\
\exists b \forall x \forall y (x \in X \land y \in Y \rightarrow Bbxy \land b \neq x \land b \neq y)]$$

Indeed, using $A_{12}$ in conjunction with $A_i - A_{i-1}$ and Tarski’s aforementioned representation theorem one may readily prove

**Theorem 19** [Ehrlich 1987]. A structure $\langle A, B, \equiv \rangle$ is a model of absolutely continuous elementary Euclidean geometry (i.e., $A_i - A_{i-1}$) if and only if $\langle A, B, \equiv \rangle$ is isomorphic to $C_2(\mathbb{N})$.

Using properties of $\langle \mathbb{N},+,\cdot,\langle,0,1\rangle$ and $\langle \mathbb{R},+,\cdot,\langle,0,1\rangle$ together with properties of the theory and models of $E$, one may prove the following geometrical analog of Theorem 18 relating $C_2(\mathbb{N})$ and $C_2(\mathbb{R})$, where the geometrical analogs of the concepts of a homogeneous-universal ordered field (resp. Archimedean ordered field) and of a universally extending ordered field (resp. Archimedean ordered field) are defined in the expected manner:
Theorem 20 [Ehrlich 1987; 1989a]. (I) The following sets of axioms constitute
(categorical) axiomatizations of $C_2(\mathbb{R})$; (II) by deleting the Archimedean axiom from
the following axiomatizations one obtains categorical axiomatizations of $C_2(\text{No})$.

Axioms for elementary Euclidean geometry, i.e., $E$
Archimedean axiom (appropriate for $E$ [Ehrlich 1997, p. 62])

Axiom of (Geometric) Homogeneous-Universality
or, alternatively
Axiom of (Geometric) Universal Extensibility

whereby the Axiom of Geometric Homogeneous-Universality (Axiom of Geometric
Universal Extensibility) we mean the assertion: The collection of points together with the
 corresponding relations defined on it constitutes a Homogeneous-Universal (Universally
 Extending) model of the above stated axioms.

In addition, by appealing to the homogeneity of $C_2(\text{No})$ one can establish the
amorphous nature of $C_2(\text{No})$. We hasten to emphasize, however, that there are more
revealing geometrical considerations that lead to the same conclusion regarding models
of elementary Euclidean geometry more generally. In particular, it is well-known that
every Euclidean rigid motion (products of translations, rotations and reflections)\(^{28}\) of a
model of classical Euclidean geometry is an automorphism of the model (cf. [Greenberg
1993, ch. 9; Venema 2006, ch. 12]) and that this theorem of classical geometry carries
over to arbitrary models of elementary Euclidean geometry [Hartshorne 2000, pp. 148-
158; Schwabhäuser, Szmielew, and Tarski 1983, p. 36].\(^ {29}\) The Euclidean rigid motions or

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\(^{28}\) The Euclidean rigid motions constitute a group of automorphisms, the identity
automorphism being the identity of the group. The term “product” refers to the group
operation.

\(^{29}\) In fact, as the just-cited discussion in [Hartshorne 2000] shows, these results--which
are generalizations of well-known results of classical Euclidean geometry (cf. [Greenberg
1993, ch. 9; Venema 2006, ch. 12])--apply to arbitrary Hilbert planes, which include the
planes of elementary geometry as very special cases.
isometries of a model \( \langle A, B, \equiv \rangle \) of elementary Euclidean geometry coincide with the one-to-one onto mappings \( f: A \rightarrow A \) that satisfy the condition: \( ab \equiv f(a)f(b) \) for all \( a, b \in A \); that is, they are precisely the distance preserving surjections of the plane (cf. [Hartshorne 2000, p. 155,17.2]). On the basis of this it is easy to see that every such model has as many distinct nontrivial automorphisms as there are points of the space and that these nontrivial automorphisms carry with them nontrivial automorphisms of the model’s substructures whose universes consist of the points of a line thereof.

Some indication of the range of nontrivial isometries of a model \( \langle A, B, \equiv \rangle \) of \( E \) is given by the following well-known result that applies to models of \( A_i - A_{i0} \) more generally (cf. [Hartshorne 2000, ibid]):

i. For any pair of points \( a, a' \in A \), there is an isometry \( f \) such that \( f(a) = a' \).

ii. For any three points \( o, a, a' \in A \), there is an isometry \( f \) such that \( f(o) = o \) and \( f \) maps the ray \( \overrightarrow{oa} \) onto the ray \( \overrightarrow{o'a} \).

iii. For any line \( L \) of \( \langle A, B, \equiv \rangle \), there is an isometry \( f \) such that \( f(a) = a \) for all \( a \in L \) and \( f \) interchanges the points on the two sides of \( L \).

As the qualifying phrase “some indication” suggests, however, the isometries referred to in i-iii above do not exhaust the isometries of \( \langle A, B, \equiv \rangle \); nor, for that matter, do the nontrivial isometries exhaust the nontrivial automorphisms of \( \langle A, B, \equiv \rangle \). Nevertheless, 

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30 A more general class of automorphisms of \( \langle A, B, \equiv \rangle \) is the similarities, that is, the one-to-one onto mappings \( f: A \rightarrow A \) that satisfy the condition: \( f(a)f(b) \equiv kab \) for all \( a, b \in A \) where \( k \) is a positive element of the characteristic field of \( \langle A, B, \equiv \rangle \). These automorphisms are called “similarities” because they map every triangle of a plane onto a similar triangle. Traditionally, however, Euclidean geometry has been associated with isometries as opposed to similarities since “size” is considered an invariant property of Euclidean space. This is intimately related to the property of “free mobility” associated (albeit not exclusively) with Euclidean geometry, and has its roots in the traditional Euclidean proof-technique of superimposing congruent Euclidean figures on one another. Indeed, it was with this in mind that Felix Klein, as part of his Erlanger Program, identified Euclidean geometry as the study of the those properties—incidence, betweenness, distance, etc.—that remain invariant under its group of isometries.
on the basis what has been said it is evident that, unlike the ordered field of real numbers and the s-hierarchical ordered field of surreal numbers, the systems of points comprising the models of elementary Euclidean geometry are highly amorphous indeed.

Despite the disparity, the idea of a Cartesian space over an ordered number field does provide the requisite vehicle for bridging the gap between the heterogeneous domains. Indeed, much as the Cartesian space over the ordered field of real numbers succeeds in bridging the gap between the domains of number and of classical Euclidean geometry, the Cartesian space over the ordered field of surreal numbers bridges the gap between the domains of number and of absolutely continuous elementary Euclidean geometry. Underlying this success is the fact that the analytic representations afforded by these structures are intrinsically amorphous, and rightfully so, even when the ordered number fields in question or their corresponding s-hierarchical counterparts are not. It is this unsung feature of the classical scaffolding that we believe is central to the resolution of the age-old problem recounted by Fraenkel, Bar-Hillel and Levy in the opening passages of our text.

REFERENCES


Alling, Norman, and Ehrlich, Philip: 1987, Sections 4.02 and 4.03 of Alling [1987].


While distance has been traditionally regarded as an invariant of Euclidean space, it is sometimes important to consider the affine structure \( \langle A, B \rangle \) of a model of \( E \) separately. For the automorphisms of \( \langle A, B \rangle \), see [Szmielew 1893].


