Our textbook introduces the notion of distance in the plane on page 12 of chapter 1. In this group work exercise you and your team will explore this notion in greater depth. In all of your group work you should strive to develop the ability to reason from concepts. Reasoning from concepts is very different from the “memorize a formula” strategy. It may take a lot of work but you will eventually see the benefits of reasoning from concepts. Mathematicians argue and disagree about many things, but every mathematician would agree that reasoning from concepts leads to long term mastery. You will just have to trust me (us) that it is true and in your best interest.

**Objective:** In a series of three steps you will deduce the formula for an arbitrary point along a line segment in cartesian coordinates. You will do this by using what you learned in high school geometry or perhaps you will extend your knowledge of geometry. In future episodes of group work you will come to rely less and less on my roadmap and begin more and more to build your own. It is possible that your group will discover other solutions on your own. You are strongly encouraged to explore your own path. You will receive extra credit for any original work. But, please make sure that you complete the steps as I have outlined them for full credit on this assignment.

1. Find a formula for the midpoint of an interval on the real line.
2. Prove Thales’s intercept theorem using concepts from Euclidean plane geometry.
3. Deduce a formula for the coordinates of the midpoint of an arbitrary line segment in the Cartesian plane.
Key Concepts:

**Length:** The length of an interval \((a, b) \subset \mathbb{R}\), is \(b - a\). Notice that there is no need for absolute values here. Why? The lengths of \([a, b)\), \((a, b]\) and \([a, b]\) are all also equal to \(b - a\).

**Distance:** In one dimension the distance between two points and the length the interval between them are the same thing. Does this make sense? In the two dimensional plane distance is defined through the Pythagorean theorem: \(\alpha^2 + \beta^2 = \gamma^2\)

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  γ
 / \\
/    \\
  β
/     \\
α
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**Midpoint:** A midpoint is the unique point that is equidistant from two endpoints.

**Parallel Projection:** Hopefully you have already studied reflections. Points in the plane are reflected about a line via perpendicularks. Projections are a related concept. Projections can carry points in the plane such as \(a, b, c\), from one line, \(l_1\), to points \(a', b', c'\), on another line \(l_2\). The key property of parallel projection is that it preserves proportions (as you will prove).
**Assignment:** After today’s class please spend 24 hrs of quality brain time engaging with the assignment on your own. Record your thoughts, questions and scratch work. Then as you work freely with your teammates update your notes and ideas based on your conversations. Please keep an annotated record of your meetings (date and time) and progress. After you have checked your work and written your final solution please take some time for honest reflection about your experience. The final component of your work is to write a reflection. In your reflection you should write about what worked for you and what did not, but most importantly ask and answer: why? Each of you will turn in three documents for full credit:

- Scratchwork with Annotations.
- Formal Solution.
- Written Reflection.

**Step 1:** Start by showing that the length of an interval is additive: Suppose $a < b < c$, how can you express the length of the interval $(a,c)$ in terms of the lengths of the intervals $(a,b)$ and $(b,c)$? Use this idea of additivity together with the definition of midpoint to deduce a formula for the midpoint of an interval $(a,b)$. What about a point that is five sixths of the way?

**Step 2:** Use the notion of similar triangles to prove that if the lines $\overrightarrow{aa'}, \overrightarrow{bb'}$ and $\overrightarrow{cc'}$ are parallel then

$$\frac{ab}{ac} = \frac{a'b'}{a'c'}$$

Use this result to conclude that if $b$ is the midpoint of $\overrightarrow{ac}$ then $b'$ must be the midpoint of $\overrightarrow{a'c'}$.

**Step 3:** Pick any two points in the Cartesian plane $P(x_0, y_0)$ and $Q(x_1, y_1)$. Draw the line segment between them. Let M be the midpoint of this segment. To find the coordinates of the point M perform a parallel projection on the $x$-axis and also the $y$-axis. Use the results of Step 1 and 2 above to find a formula for the midpoint. Now suppose that N is a point that is five sixth’s of the way from P to Q. What are the coordinates of N?