MATH 1350 (Barsamian) Day 16  Monday Sept 19, 2016

- Sign In

- Quiz 4 this coming Friday, Sept 23

Section 3.1  The Constant

Bank account interest

Simple Interest

Deposit money into account.

Account earns interest only on the amount of money that you deposited. As interest accumulates, you don’t earn interest on the interest. You only earn interest on the original amount.
Simple Interest Formulas

\[ P = \text{principal} = \text{amount originally deposited} \]
\[ r = \text{interest rate expressed as a decimal (not as a percentage)} \]
\[ t = \text{number of years the money is in the account} \]
\[ A = \text{the amount of money in the account at time } t. \]

\[ A = P + Prt = P(1 + rt) \]

Principal \[\uparrow\]
earned interest

factored form
Graph $A$ as a function of time

$A = (P \cdot r) \cdot t + P$ will be line with slope $m = Pr$ and y-intercept $(0, P)$

Example

Deposit $1000$ into account that has $3\%$ simple interest. What will be the balance after 7 years?

Solution

$P = 1000$, $r = 0.03$, $t = 7$, $A$ unknown (find $A$)

$A = 1000 \left( 1 + 0.03 \cdot 7 \right) = 1000 \left( 1 + 0.21 \right) = 1000(1.21) = 1210$
Periodically Compounded Interest

Deposit money into account.

It earns simple interest for a while.

At some point in time, the earned interest gets added to the principal, and you start earning simple interest again, but now earning interest on the new, larger amount of money. This gets repeated periodically.
Graph of Periodically-Compounded Interest

Periodically-compounded Interest Formula:

\[ A = P \left( 1 + \frac{r}{m} \right)^{m \cdot t} \]

Note: The formula is only valid at the times where the interest is being compounded.

\[ m = \text{number of times per year that the interest gets compounded} \]
Example #2: Deposit $1000 into account with 3% interest compounded yearly. What will be the balance after 7 years?

Solution: \( P = 1000 \), \( r = 0.03 \), \( t = 7 \), \( m = 1 \)

\[
A = 1000 \left(1 + \frac{0.03}{1}\right)^{7} \approx 1229.67
\]

Example #3: Same numbers, but now compound monthly \( m = 12 \)

\[
A = 1000 \left(1 + \frac{0.03}{12}\right)^{12 \cdot 7} \approx 1233.35
\]

Example #4: Same numbers, but now compound daily \( m = 365 \)

\[
A = 1000 \left(1 + \frac{0.03}{365}\right)^{365 \cdot 7} \approx 1233.62
\]
Notice the balances

\[ 1210 < 1229.87 < 1233.35 < 1233.67 \]

Simple Interest, \[ m = 1 \] 
Never compounded \[ m = 365 \]

These numbers are definitely increasing, but they are increasing less and less as \( m \) gets bigger and bigger.

They seem to be leveling off at a number near \$ 1233.

Obvious question: what is the exact number that these balances are approaching?

That is \( \lim_{m \to \infty} P(1 + \frac{r}{m})^m \) ??
Related Simpler Question: What is \( \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \)?

Investigate this first. Use a table of values:

<table>
<thead>
<tr>
<th>(n)</th>
<th>Value of (\left(1 + \frac{1}{n}\right)^n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n=1)</td>
<td>(\left(1 + \frac{1}{1}\right)^1 = 2^1 = 2)</td>
</tr>
<tr>
<td>(n=10)</td>
<td>(\left(1 + \frac{1}{10}\right)^{10} \approx 2.59374)</td>
</tr>
<tr>
<td>(n=100)</td>
<td>(\left(1 + \frac{1}{100}\right)^{100} \approx 2.70481)</td>
</tr>
<tr>
<td>(n=1000)</td>
<td>(\left(1 + \frac{1}{1000}\right)^{1000} = (1.001)^{1000} \approx 2.71692)</td>
</tr>
<tr>
<td>(n=10000)</td>
<td>(\left(1 + \frac{1}{10000}\right)^{10000} \approx 2.71815)</td>
</tr>
</tbody>
</table>

It seems like the limit exists and is a number close to 2.718.


**Big Facts** From higher math we know

- the limit does exist. It is given the symbol $e$.

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

*real number*

**Facts about $e$**

- it is a real number close to $2.718$
- it is *irrational*
  - It cannot be expressed exactly as a fraction or as a terminating or repeating decimal.
Related limits
\[ \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x \]

the natural exponential function

\[ \lim_{m \to \infty} P \left(1 + \frac{r}{m}\right)^{m \cdot t} = P e^{r \cdot t} \]

This answers our question from above.

Inspired by this, we invent a new kind of bank account, where the balance is computed using the formula
\[ A = P e^{r \cdot t} \]

Called continuously compounded interest.
Example: Deposit $1000 into account with 3% interest compounded continuously. Find the balance after 7 years.

Solution: \( P = 1000, \ r = 0.03, \ t = 7 \)

\[ A = 1000 \cdot e^{(0.03 \cdot 7)} \approx $1233.68 \]

Notice:

\[ 1210 < 1229.87 < 1233.35 < 1233.67 < \cdots < 1233.68 \]

Simple Interest Never Compounded

\( M=1 \quad M=12 \quad M=365 \)

Continuously Compounded