Section 5.4 The Fundamental Theorem of Calculus

**Area Function.**
Given a continuous function $f(t)$

The symbol $\int_{t=2}^{t=7} f(t) \, dt$ represents the number that is the signed area under graph of $f$ from $t=2$ to $t=7$

The symbol $\int_{t=2}^{t=5} f(t) \, dt$ is just the area from $t=2$ to $t=5$
The symbol \( \int_{t=2}^{t=x} f(t) \, dt \) is the area from \( t=2 \) to \( t=x \)

where \( x \) is unknown.

So the expression \( \int_{t=2}^{t=x} f(t) \, dt \) represents a function of the variable \( x \). Once a value for \( x \) is chosen, then the function spits out an output number.

Call this function \( g(x) = \int_{t=2}^{t=x} f(t) \, dt \) an "Area Function".

notice: the variable is \( x \), not \( t \)

a cool property: \( g(2) = 0 \)
The Fundamental Theorem of Calculus

If \( g(x) = \int_{t=a}^{t=x} f(t) \, dt \) then \( g(x) \) is an antiderivative of \( f(x) \). That is \( g'(x) = f(x) \).

Single equation form

\[
\frac{d}{dx} \left( \int_{t=a}^{t=x} f(t) \, dt \right) = f(x)
\]

Example: Let \( f(x) = 2x + 3 \)

Then \( g(x) = \int_{t=1}^{t=x} 2t + 3 \, dt \) is an antiderivative of \( f(x) \).

So \( g'(x) = \frac{d}{dx} \left( \int_{t=1}^{t=x} 2t + 3 \, dt \right) = 2x + 3 \)
It may seem silly to write an antiderivative as an integral. In the current example, we can write \( g(x) \) as a regular function.

\[
g(x) = \int_{t=1}^{t=x} 2t+3 \, dt = \frac{2x+3}{2} + C.
\]

At \( t=1 \):

\[
(1,5)
\]

At \( t=x \):

\[
(x,2x+3)
\]

\[
\frac{2x+3}{2} + C = \frac{1}{2} (x-1)(2x-2)
\]

\[
= 5(x-1) + \frac{1}{2} (x-1)(2x-2)
\]

\[
= 5x - 5 + (x-1)(x-1)
\]

\[
= 5x - 5 + (x-1)^2
\]

\[
= 5x - 5 + x^2 - 2x + 1
\]

\[
g(x) = x^2 + 3x - 4
\]
notice: \( g'(x) = \frac{d}{dx}(x^2 + 3x - 4) = 2x + 3 = f(x) \)

\( g(1) = 1^2 + 3(1) - 4 = 0 \)

Harry says this is the antiderivative of \( f(x) \)

\[ h(x) = \int_{t=2}^{t=x} 2t + 3 \, dt \]

\[ h(x) = (x-2)(x-2) \frac{1}{2} (x-2)(2x-4) \]

\[ = (x-2)(x-2) \frac{1}{2} (2x-4) \]

\[ = 7x - 14 + \frac{1}{2}(x-2)(x-2) \]

\[ = 7x - 14 + (x-2)(x-2) = 7x - 14 + x^2 - 4x + 4 \]

\[ h(8) = x^2 + 3x - 10 \]
Notice: \( h'(x) = 2x + 3 = f(x) \)

\( h'(\theta 2) = \frac{2}{4} \left( 2^2 + 3(2) \right) - 10 = 4 + 6 - 10 = 0 \)

\[ g(x) = \int_{t=1}^{t=x} (2t^2 + 3t + c) \, dt \]

\[ = \left[ \frac{(t^2 + 3t + c)^2}{2} \right]_{t=1}^{t=x} \]

\[ = \left( x^2 + 3x + c \right) - \left( 1^2 + 3(1) + c \right) \]

\[ = x^2 + 3x - 4 \quad \text{(same expression)} \]

We have easier ways of setting a normal function form for \( g(x) \) and \( h(x) \)
But why do this? Why write \( g(x) = \int_{t=1}^{t=x} 2t+3 \, dt \) for an antiderivative of \( f(x) = 2x+3 \)?

When you could write \( g(x) = x^2 + 3x - x \) instead??

(Good Question!) For the answer, consider this next example:

Another example:

\[
\text{let } g(x) = \int_{t=0}^{t=x} e^{-t^2} \, dt
\]

So what is \( g'(x) \)?

Answer \( g'(x) = \frac{d}{dx} \left( \int_{t=0}^{t=x} e^{-t^2} \, dt \right) = e^{(-x^2)} \)

So \( g(x) \) is an antiderivative of \( f(x) = e^{(-x^2)} \)

Big fact: \( g \) can't be written as a regular function!!
New topic

Question: Given a function $f(x)$ continuous on an interval $[a, b]$.

Question: How high must a rectangle on the same interval need to be to enclose the same area?

Answer: Required area on left = area on right

\[
\int_{x=a}^{x=b} f(x) \, dx = (b-a) \cdot h
\]
So \( h = \frac{1}{b-a} \int_{x=a}^{x=b} f(x) \, dx \)

Defined to be the average value of \( f(x) \) over the interval \([a, b]\)

Example

Find average value of \( f(x) = \cos^2(x) \) over the interval \([-\pi, \pi]\)

Solution

\[
h = \frac{1}{\pi - (-\pi/2)} \int_{x=-\pi/2}^{x=\pi/2} \cos^2(x) \, dx
\]

\[
= \frac{1}{\pi} \left( \sin(\pi x) \right) \bigg|_{x=-\pi/2}^{x=\pi/2} = \frac{1}{\pi} \left( \sin(\pi) - \sin(-\pi/2) \right)
\]

\[
= \frac{1}{\pi} \left( 1 - (-1) \right) = \frac{2}{\pi}
\]
$y = \cos(x)$

$\left(-\frac{\pi}{2}, 0\right)$

$\left(\frac{\pi}{2}, 0\right)$

$h = \frac{2}{n}$

end of lecture