Tangent Line Example

For the function $f(x) = -x^2 + 8x - 7$ (function from Monday's Examples)

Find the equation for the line tangent to the graph of $f$ at $x=2$. (Present the line equation in slope-intercept form.)
Tangent Line Background

- Point slope form for the equation of a line that has known point \((a, b)\) and known slope \(m\):
  \[
  (y - b) = m(x - a)
  \]

- Stuff that we know about the line tangent to graph of \(f\) at \(x = a\):
  - It passes through the known point \((a, f(a))\)
  - It has slope \(m = f'(a)\)

- So the point slope form for the line equation for the line tangent to graph of \(f\) at \(x = a\):
  \[
  (y - f(a)) = f'(a)(x - a)
  \]

So if we are asked for equation of a tangent line, this is what we build.
Return to our Example \( f(x) = -x^2 + 8x - 7 \)

We want equation for the line tangent to graph of \( f \) at \( x = 2 \).

Solution: We need to build this \( (y - f(a)) = f'(a)(x-a) \)

Get Parts

Identify \( a \): \( a = 2 \) (this is the x-coordinate of point of tangency)

Find \( f(a) \): \( f(2) = -(2)^2 + 8(2) - 7 = -4 + 16 - 7 = 5 \) (this is the y-coordinate of point of tangency)

Find \( f'(a) \): \( f'(2) = 4 \) (result from Monday)

Assemble Parts

\[(y - 5) = 4(x - 2)\]
Now convert to slope intercept form

\[ y - 5 = 4(x - 2) \]

\[ y - 5 = 4x - 8 \]

\[ y = 4x - 3 \]  \text{Tangent line equation}

Example similar to 2.1 #18 (suggested exercise)

The line tangent to graph of some function \( g(x) \) at the point \( (3, 7) \) passes through the point \( (5, 18) \)

a) find \( g(3) \)

b) find \( g'(3) \)
Solution:
The fact that the line is tangent to graph of \( g(x) \) at \((3,7)\) tells us that \((x,y) = (3,7)\) is on the graph of \( g \).

\[ g(3) = 7. \]

\[ g'(3) = \text{slope of the tangent line} = \frac{\Delta y}{\Delta x} = \frac{18-7}{5-3} = \frac{11}{2} = 5.5 \]
Example similar to 2.1 #6

Find equation of the line tangent to graph of \( h(x) \) at \( x = -2 \) if \( h(-2) = 13 \) and \( h'(-2) = -5 \)

Solution

We need to build this

\[(y - h(a)) = h'(a)(x-a)\]

Parts: \( a = -2 \)

\[h(a) = h(-2) = 13\]
\[h'(a) = h'(-2) = -5\]

Assemble Parts

\[(y - 13) = -5(x + 2)\]

Convert to slope-intercept form

\[y - 13 = -5(x + 2) = -5x - 10\]
\[y = -5x + 3\]
Harder example for \( f(x) = \frac{10x}{6 + x^2} \)

Find equation of the line tangent to graph of \( f \) at \( x = 3 \)

**Solution** We need to build \((y - f(a)) = f'(a)(x-a)\)

Get parts

\( a = 3 \)

\[
f(a) = f(3) = \frac{10(3)}{6 + (3)^2} = \frac{30}{6+9} = \frac{30}{15} = 2
\]

\[
f'(a) = f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} \quad \text{Definition of Derivative}
\]
To build the expression for $f'(3)$ we will need to know what $f(3+h)$ is. Figure that out first:

$$f(x) = \frac{10x}{6+x^2}$$

$$f(\ ) = \frac{10(\ )}{6+(\ )^2}$$  \hspace{1cm} \text{empty version of } f

Substitute $3+h$ into the parentheses

$$f(3+h) = \frac{10(3+h)}{6+(3+h)^2}$$

Now use this $f(3+h)$ in the formula for $f'(3)$. 
\[ \frac{f'(3)}{h} = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} \]

\[ = \lim_{h \to 0} \frac{\frac{10(3+h)}{b + (3+h)^2} - 2}{h} \]

\[ = \lim_{h \to 0} \frac{1}{h} \left( \frac{10(3+h)}{b + (3+h)^2} - 2 \right) \]

\[ = \lim_{h \to 0} \frac{1}{h} \left( \frac{10(3+h)}{b + (3+h)^2} - 2 \left( \frac{b + (3+h)^2}{b + (3+h)^2} \right) \right) \]

\[ = \lim_{h \to 0} \frac{1}{h} \left( \frac{30 + 10h - 12 - 2(9 + 6h + h^2)}{b + (3+h)^2} \right) \]

\[ = \lim_{h \to 0} \frac{1}{h} \left( \frac{30 + 10h - 12 - 2(9 + 6h + h^2)}{b + (3+h)^2} \right) \]
\[
\lim_{h \to 0} \frac{1}{h} \left( \frac{30 + 10h - 12 - 18 - 12h - 2h^2}{6 + (3+3h)^2} \right)
\]

the red terms cancel
the green terms combine

\[
= \lim_{h \to 0} \frac{1}{h} \left( \frac{-2h - 2h^2}{6 + (3+h)^2} \right)
\]

factor out $-2h$

\[
= \lim_{h \to 0} \frac{-2h}{h} \left( \frac{1 + h}{6 + (3+h)^2} \right)
\]

Since $h \to 0$, we know $h \neq 0$
so we can cancel

rational function with $h = 0$ in domain

So can substitute $h = 0$

\[
= \frac{-2(1)}{6 + (3)^2} = \frac{-2}{6 + 9} = \frac{-2}{15} = m = f'(3)
\]
Assemble parts

\[(y - 2) = -\frac{2}{15} (x - 3)\]

Convert to slope intercept form

\[y - 2 = \left(-\frac{2}{15}\right)(x - 3) = \left(-\frac{2}{15}\right)x - \frac{2}{15}(-3)\]

\[y - 2 = \left(-\frac{2}{15}\right)x + \frac{2}{5}\]

\[y = \left(-\frac{2}{15}\right)x + \frac{2}{5} + 2\]

\[y = \left(-\frac{2}{15}\right)x + \frac{12}{5}\]

Slope intercept form of the equation of the tangent line

End of Lecture