Section 6.2 Total Income for Continuous Income Stream
Future Value for Continuous Income Stream

Not covering probability distributions, pages 421, 422

Total Income For A Continuous Income Stream

Imagine a bucket of money

Money flowing into the bucket with flow rate \( f(t) \)
Questions will be about the amount of money in the bucket, $A(t)$.

Notice that

$$A'(t) = f(t)$$

rate of change of the amount of money in the bucket = flow rate of the incoming flow of money.

**Simplest Examples have Constant Flow**

**Example 1:** Find the Total Income produced by a continuous income stream in the first 10 years if the flow rate is $f(t) = 3000$ dollars per year. (constant flow)

**Solution**

Total income = $\Delta A = A(10) - A(0) = \int_{t=0}^{t=10} A'(t) \, dt = \int_{t=0}^{t=10} 3000 \, dt$

Because $A'(t) = f(t)$
\[\begin{align*}
&= (3000t + C) \bigg|_{t=0}^{t=10} \\
&= (3000(10) + C) - \frac{3000(0) + C}{\text{zero}} \\
&= 3000 \times 10 \text{ dollars} \\
\Delta A &= 30,000 \text{ dollars} \\
&\text{Illustrate with a graph}
\end{align*}\]
Example #2

Find the total income produced by a continuous income stream in the first two years if the flow rate is \( f(t) = 600e^{(0.06t)} \) not a constant flow!

Solution

Total Income = \( \Delta A = A(2) - A(1) = \int_{t=0}^{t=2} A'(t)\,dt = \int_{t=0}^{t=2} f(t)\,dt \)

\[ = \int_{t=0}^{t=2} 600e^{(0.06t)}\,dt = 600 \int_{t=0}^{t=2} e^{(0.06t)}\,dt \]

\[ = 600 \left( \frac{e^{(0.06t)}}{0.06} + C \right) \bigg|_{t=0}^{t=2} \]

\[ = 600 \left( e^{(0.06 \cdot 2)} \right) \bigg|_{t=0}^{t=2} \]

(The + C will cancel)
\[
\begin{align*}
M &= 10000 \ e^{(0.06t)} \bigg|_{t=2}^{t=0} \\
&= 10,000 \ (e^{0.06(2)}) - e^{(0.06)(0)} \\
&= 10,000 \ (e^{0.12} - e^0) \\
&= 10,000 \ (e^{0.12} - 1) \\
&\approx \$1275
\end{align*}
\]
Future Value Problems: Money flowing into a Bucket

Added Twist: The money that is already in the bucket is earning interest.

flow rate \( f(t) \)

Money in bucket earning interest

(Basically, money is flowing into an account where it will start earning interest as soon as it arrives.)

Complication: Money that flows into the bucket early will have a long time to earn interest.

Money that flows in later won't have as much time to earn interest.
Formula for Future Value of a Continuous Income Stream

\[ FV = \int_{t=0}^{T} f(t) e^{r(T-t)} \, dt = e^{rT} \int_{t=0}^{T} f(t) e^{-rt} \, dt \]

- **T** = constant representing ending time
- **f(t)** = flow rate of money coming in
- \( e^{rT} \) = continuously-compounded interest rate earned by the money that has flowed in.
- **FV** = value at the amount of money in the account at the future time \( t = T \).
Example Similar to suggested exercise 6.2 #43

Find future value, at 2.95% interest compounded continuously for 6 years, of a continuous income stream with flow rate \( f(t) = 2000e^{0.06t} \)

Solution Build FV expression

\[ T = 6 \]
\[ r = 0.0295 \]
\[ f(t) = 2000e^{0.06t} \]

\[ FV = e^{0.0295}(6) \int_{0}^{6} 2000 e^{0.06t} e^{-0.0295t} \, dt \]

\[ = e^{0.0295}(6) \cdot 2000 \int_{0}^{6} e^{0.06t - 0.0295t} \, dt \]
\[
\begin{align*}
&= e^{0.295/6} \cdot 2000 \int_{t=0}^{t=6} e^{0.0305t} dt \\
&= e^{0.295/6} \cdot 2000 \left( \frac{e^{0.0305\cdot6} + C}{0.0305} \right)_{t=0}^{t=6} \quad \text{where} \quad 0.06t - 0.295 = 0.0305t \\
&= e^{0.295/6} \cdot 2000 \left( \frac{e^{0.0305\cdot6} - e^{0.0305\cdot0}}{0.0305} \right) \\
&= e^{0.295/6} \cdot 2000 \cdot \frac{1}{0.0305} \left( e^{0.0305\cdot6} - 1 \right) \approx \$15717.92 \\
\end{align*}
\]

\text{Exact} \quad $15717.92$
Remark: We can also use a computer to find the definite integral

\[ FV = e^{(.0295)(6)} \int_{0}^{6} 2000 e^{(.06t)} e^{(-.0295t)} dt \]

Using Wolfram Alpha

\[ \approx 15,717^{\text{^2}} \]

end of lecture