Remember: Exam 4 is next Wednesday, April 13

Section 5.5

Our preliminary definition of the Definite Integral

\[ \int_{x=a}^{x=b} f(x) \, dx = \text{signed area under graph of } f(x) \text{ from } x=a \text{ to } x=b. \]

only worked when graph of \( f \) was made up of basic shapes whose area we could compute using geometry.

For more general graphs, we don’t have geometric formulas for the area. We would like to have some definition of what we even mean by the area of a general curvy region.

Wednesday, we arrived at this new definition of Definite Integral

\[
\text{Definite Integral of } f \text{ from } x=a \text{ to } x=b = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x = \lim_{n \to \infty} L_n = \lim_{n \to \infty} R_n
\]

Definition \quad \text{Fact that these are equal}
We will define area of a general curvy region as follows:

Signed area under graph of \( f \) from \( x = a \) to \( x = b \)

\[
\int_{x=a}^{x=b} f(x) \, dx = \lim_{N \to \infty} L_N
\]

We want an analytic way to compute \( \text{SA} = \int_{x=a}^{x=b} f(x) \, dx \) = \( \lim_{x \to \infty} \ln \).

(We have guessed a limit of this sort using a computer, on Wednesday. We guessed an estimate.)
Wish List: We would like to find these three signed areas.

#1 \[ SA = \int_{x=2}^{x=12} \left( 5 + \frac{x^2}{10} \right) \, dx \]

#2 \[ SA = \int_{x=0}^{x=2} 4e^{(x)} \, dx \]

#3 \[ SA = \int_{x=1}^{x=5} \frac{2}{x} \, dx \]

In all three examples, we don't have a geometric formula for the area.

What do we do?
If we set up the mathematical expressions for \( \ln \) in each case, and then try to find the limit \( \lim_{n \to \infty} \ln \n\), we encounter math beyond the level of this course.

(Not a lot above the level of this course. Junior/Senior level math classes for math majors do this kind of thing.)

That brings us to section 5.5.
Section 5.5 The Fundamental Theorem of Calculus

We have seen two uses of the integral symbol $\int$

1. The indefinite integral $\int f(x)dx$ has to do with antiderivatives of the function $f(x)$.

2. The definite integral $\int_{a}^{b} f(x)dx$ has to do with area.

These two concepts (antiderivative and area) seem unrelated.

It seems strange that such similar-looking symbols would be used to denote such different concepts.
But it turns out that there is a very important relationship between the antiderivatives of \( f \) and the area under graph of \( f \).

Here is the relationship:

- Given a function \( f(x) \) and an interval \([a, b]\)
  - The **indefinite integral** \( \int f(x) \, dx \) can be used to find an antiderivative \( F(x) \).
  - This is a function.

- The **definite integral** \( \int_{x=a}^{x=b} f(x) \, dx \) represents a number that is a signed area, \( SA \).

Here is the relationship:

\[
SA = \int_{x=a}^{x=b} f(x) \, dx = F(b) - F(a)
\]
Let's apply the Fundamental Theorem to our three Wish List Examples from page (3)

Example #1  Find $SA = \int_{x=2}^{x=12} \frac{5 + x^2}{10} \, dx$

Solution

$f(x) = 5 + \frac{x^2}{10}$

$F(x) = \int f(x) \, dx = \int 5 + \frac{x^2}{10} \, dx = \int 5 \, dx + \frac{1}{10} \int x^2 \, dx$

$= 5x + \frac{1}{10} \left( \frac{x^3}{3} \right) + C$

$= 5x + \frac{x^3}{30} + C$

$F(6) = F(12) = 5(12) + \frac{12^3}{30} + C = 60 + \frac{1728}{30} + C$

$= 117.6 + C$

$F(2) = F(2) = 5(2) + \frac{2^3}{30} + C = 10 + \frac{8}{30} + C$

$= 10.266 + C$
So \( F(61) - F(9) = (117.6 + c) - (10.266 + c) \)
\[ = 107.334 \]

This is about what we guessed on Wednesday!!

Conclusion \[ \int_{x=2}^{x=12} \frac{x^2 + 5}{x^3} \, dx = 107.333 \]

Example #2 \[ \int_{x=0}^{x=2} 4e^{6x} \, dx \]

\[ f(x) = 4e^{6x} \]
\[ F(x) = \int 4e^{6x} \, dx = \frac{1}{6} 4e^{6x} + C \]
\[ S_A = \int_{x=0}^{x=2} 4e^{6x} \, dx = F(2) - F(0) = F(2) - \left( \frac{1}{6} 4e^{0} + C \right) \]
\[ = \left( 4e^{(2)} + C \right) - \left( 4e^{(0)} + C \right) \]
\[ = 4e^2 - 4.1 \]
\[ = 4e^2 - 4 \]
Example 3: \( \int_{x=1}^{x=5} \frac{2}{x} \, dx \)

Solution:

\( f(x) = \frac{2}{x} \)

\( F(x) = 2 \ln |x| + C \)

\( S_A = \int_{x=1}^{x=2} \frac{2}{x} \, dx = \left[ 2 \ln |x| + C \right]_{x=1}^{x=2} \)

\( = (2 \ln |2| + C) - (2 \ln |1| + C) \)

\( = 2 \ln(2) - 0 \)

\( = 2 \ln(2) \)

End of Lecture