One more Absolute Max/Min problem (Section 4.5)

For \( f(x) = \frac{20 - 4x - 250}{x^2} \)

Find all absolute extrema on the interval \((0, \infty)\)

\[\text{Solution}\]

Domain: Not a closed interval.

So can't use closed interval method.

We're not guaranteed that any absolute extrema even exist.

We have no intuition about the shape of the graph because \( f(x) \) is an unfamiliar form.

If there are any absolute extrema, they would have to occur at x-values that are critical numbers in the interval \((0, \infty)\)
So start by finding **critical numbers for f**.

Look for all \( x = c \) such that

- \( f'(c) = 0 \) or \( f'(c) \) DNE
- \( f(c) \) exists

Start by finding \( f'(x) \)

\[
f(x) = 20 - 4x - \frac{250}{x^2}
\]

\[
f'(x) = \frac{d}{dx} \left( 20 - 4x - 250x^{-2} \right)
\]

\[
= 0 - 4 \frac{d}{dx} x - 250 \frac{d}{dx} x^{-2}
\]

\[
= (0 - 4(1)) - 250(2x^3)
\]

\[
f'(x) = -4 + \frac{250(2)}{x^3}
\]
Are there any x-values that cause \( f'(x) \) to not exist? Yes, \( x = 0 \).

Are there any x-values that cause \( f'(x) = 0 \)?

Set \( f'(x) = 0 \) and solve for \( x \).

\[
0 = f'(x) = -4 + \frac{250(2)}{x^3}
\]

Add 4 to both sides:

\[
4 = \frac{250(2)}{x^3}
\]

Multiply both sides by \( x^3 \):

\[
4x^3 = 250(2)
\]

Divide by 4:

\[
x^3 = \frac{250(2)}{4} = 125
\]

\[
x = 5 \quad \text{check } f'(5) = -4 + \frac{250(2)}{(5)^3} = -4 + \frac{250(2)}{125}
\]

Partition numbers for \( f' \) are \( x = 0, x = 5 \).
Are these numbers critical numbers for $f$?

Notice $f'(0)$ DNE but $f'(5)$ does exist.

So $x=5$ is the only critical number for $f$.

Determine what happens at $x=5$.

Do this by two methods.

**Method 1** Study the sign of $f'$, use it to determine behavior of $f$.

**Sign Chart for $f'(x) = -4 + \frac{250(2)}{x^3}$**

<table>
<thead>
<tr>
<th>$f'$</th>
<th>$f'$ DNE</th>
<th>$f'$ pos</th>
<th>$f'$ neg</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

Sample $x = -1$

$X = 0$

Sample $X = 1$

Sample $X = 5$

Sample $X = 10$

$f'(-1) = -4 + \frac{250(2)}{(-1)^3} = -4 + \frac{500}{-1} = \text{Neg + neg = neg}$

$f'(5) = -4 + \frac{250(2)}{(5)^3} = -4 + \frac{500}{125} = \text{pos}$

$f'(10) = -4 + \frac{250(2)}{(10)^3} = -4 + \frac{500}{1000} = -4 + \frac{1}{2} = \text{neg}$
Clearly $f$ has local max at $x=5$.
But that local max is clearly also an absolute max.

The value of the max is

$$f(5) = 20 - 4(5) - \frac{250}{(5)^2}$$

$$= 20 - 20 - \frac{250}{25}$$

$$= -10$$

Absolute max of $y = -10$ (occurs at $x = 5$)

Done with the problem
Now go back and determine behavior at \( x = 5 \).

Using Method 2, study behavior of \( f''(x) \).

Use that to determine behavior of \( f(x) \).

We have:

\[
f'(x) = -4 + 500x^{-3} = -4 + \frac{500}{x^3}
\]

\[
f'(x) = -4 + 250(2)x^{-3} = -4 + \frac{250(2)}{x^3}
\]

\[
f''(x) = \frac{d}{dx} \left( -4 + 250(2)x^{-3} \right) = 0 + 250(2) \frac{d}{dx} x^{-3}
\]

\[
= 250(2)(-3)x^{-3-1} = -250(2)(3)x^{-4}
\]

\[
f''(x) = - \frac{250(2)(3)}{x^4}
\]

On the interval \((0, \infty)\), \( x \) is positive, \( x^4 \) is positive.

So \( f''(x) = \frac{(-)(\text{pos})}{\text{pos}} = \text{negative} \).
This tells us that $f$ is concave down on the whole interval $(0, \infty)$.

We already know that at $x=5$ $f$ has a horizontal tangent line.

So $x=5$ must be the location of the absolute max. As before (in method 1), we conclude that absolute max is $y=-10$ (occurs at $x=5$).

End of method 2
End of problem.
Section 4.6 Optimization

Optimization Problems are just Absolute Max/min problems
But with possible complications

- may be word problems
- function may not be given
- domain may not be given
- may include more than one variable
Example 1
Find positive numbers $x, y$ such that
- the product is 9000
- the sum $10x + 25y$ is minimized.

Solution
Set up equations $xy = 9000$ equation 1
$S = 10x + 25y$ minimize $S$.

Use equation 1 to eliminate the variable $y$.

$x \cdot y = 9000$

$y = \frac{9000}{x}$

Substitute into equation 2

$S = 10x + 25\left(\frac{9000}{x}\right)$

This equation gives us $S$ as a function of $x$.

Use function notation $S(x) = 10x + 25\left(\frac{9000}{x}\right)$ minimize $x$. 
Domain: $x$ must be positive. $x$ in interval $(0, \infty)$

Look for critical numbers of $S$.
Start by looking for partition numbers for $S'(x)$

$$S(x) = 10x + 25 \frac{(9000)}{x}$$

Rewrite to make derivative easier

$$S'(x) = \frac{d}{dx} \left(10x + 25(9000)x^{-1}\right)$$

$$= 10 + 25(9000)(-1)x^{-1-1}$$

$$S'(x) = 10 - 25(9000)x^{-2} = 10 - \frac{25(9000)}{x^2}$$

Notice $S'(0)$ DNE so $x=0$ is partition number for $S'$. Are there any $x$-values that cause $S'(x) = 0$?

Check $0 = S'(x) = 10 - \frac{25(9000)}{x^2}$

$$\frac{25(9000)}{x^2} = 10$$

$$x^2 = \frac{25(9000)}{10} = 225000$$

$$x = \sqrt{225000}$$
\[
\frac{25(900)}{10} = x^2
\]

\[
25(900) = x^2
\]

\[
\sqrt{25 \cdot 900} = x
\]

\[
\sqrt{225 \cdot 900} = x
\]

\[
5 \cdot 30 = x
\]

\[
x = 150 \text{ special number for } f!
\]

Partition numbers for \( f' \) are \( x = 0, x = 150 \)

Critical numbers for \( f \): \( x = 150 \)

Check behavior of \( f(x) \) at \( x = 150 \)

Investigate 2nd derivative.
\[ S'(x) = 10 - 25(9000)x^{-2} \]
\[ S''(x) = 0 - 25(9000)(-2)x^{-3} = 25(9000)(2)x^{-3} \]
\[ S''(x) = \frac{25(9000)(2)}{x^3} \]

Notice when \( x \) is positive, \( x^3 \) will be positive.
So \( S'' \) will be \( \frac{\text{pos}}{\text{pos}} = \text{pos} \)
So \( S \) will be concave up on whole interval \((0, \infty)\)
So $x = 150$ is location of absolute min.

Now we need to find $y$.

Use fact that $x \cdot y = 9000$

$150 \cdot y = 9000$

$y = \frac{9000}{150} = 60$

Conclude the two numbers are $x = 150$, $y = 60$.

end of lecture