Resuming yesterday's question: Does \( \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \) exist? If so, what is its value?

**Big fact from higher math**

- The limit \( \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \) does exist.
- The value of the limit is denoted \( e \).
  
  That is \( \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e \)

- Facts about \( e \)
  - \( e \) is close to 2.718
  - \( e \) is "irrational"
    - It is not rational. \( e \) cannot be expressed exactly as a fraction, or as a terminating decimal, or even as a repeating decimal.

**Definition of the symbol \( e \)**
So the only way to re-express $e$ exactly is to use the symbol $e$.

Another related limit:

$$\lim_{{n \to \infty}} \left(1 + \frac{x}{n}\right)^n$$

**Big Fact** using change of variables, we could show that

$$\lim_{{n \to \infty}} \left(1 + \frac{x}{n}\right)^n = e^{(x)}$$

Discuss Graphs of $y = 2^x, y = 3^x, y = e^x$
Table of values

<table>
<thead>
<tr>
<th>x</th>
<th>(2^x)</th>
<th>(e^x)</th>
<th>(3^x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>(\frac{1}{4})</td>
<td>(\frac{1}{e^2})</td>
<td>(\frac{1}{9})</td>
</tr>
<tr>
<td>-1</td>
<td>(\frac{1}{2})</td>
<td>(\frac{1}{e})</td>
<td>(\frac{1}{3})</td>
</tr>
<tr>
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<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>(e)</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>(e^2)</td>
<td>9</td>
</tr>
</tbody>
</table>

Use the fact that \(2 < e < 3\)

Horizontal asymptote on left at \(y = 0\)

\((-2, \frac{1}{4})\), \((0, 1)\), \((1, e)\), \((-1, \frac{1}{3})\), \((-1, \frac{1}{2})\), \((2, 2^2)\), \((2, 3^2)\), \((2, 9)\)
Big Fact: Using more changes of variables, we could show that

\[ \lim_{m \to \infty} A = \lim_{m \to \infty} P \left(1 + \frac{r}{m}\right)^{mt} = P e^{rt} \]

The formula for the balance \( A \) for periodically-compounded bank account with principal \( P \), interest rate \( r \), compounded \( m \) times per year for \( t \) years.

Explore this formula \( P e^{rt} \)

Use \( P = 1000 \), \( r = 0.02 \), \( t = 5 \) then we get

\[ 1000 e^{0.02 \cdot 5} = 1000 e^{0.1} \approx 1105.17 \]

Exact value \( \frac{1105.17}{1} \) decimal approximation
A couple of things to notice:

The segmented graph of \( A = P(1 + \frac{r}{m})^{mt} \) from yesterday looks sort-of like the shape of the right side of an exponential graph.

And the segmented graph would look more and more like the exponential graph as \( m \to \infty \).
Also note that the formula \( A = P(1 + \frac{r}{m})^{mt} \) is difficult to use, and is only good at certain values of \( t \). (the break points, where the interest gets compounded.)

But the values given by that formula are very close to the values given by the simple expression \( Pe^{rt} \).

Inspired by this, we invent a new kind of bank account: Formula for the balance is \( A = Pe^{rt} \) (no \( m \), no limit). This kind of account is called "Continuously-compounded Interest".
Graph of $A = Pe^{rt}$ as a function of time.

Notice when $t=0$, $A = Pe^{r\cdot0} = Pe^0 = P \cdot 1 = P$.

So $(t, A) = (0, P)$ is the vertical axis intercept.
Given equation $A = Pe^{(rt)}$, involving $A, P, r, t$
Solved for $A$.

Solve for $P$

\[
\frac{A}{e^{(rt)}} = P \quad \text{Solved for } P
\]

Solve for $t$

\[
\frac{A}{P} = e^{(rt)}
\]

Take natural log of both sides

\[
\ln\left(\frac{A}{P}\right) = \ln(e^{(rt)}) = rt
\]

Divide by $r$

\[
\frac{\ln(A/P)}{r} = t \quad \text{Solved for } t
\]
Solve for \( r \)

\[ A = Pe^{rt} \]

divide by \( P \)

\[ \frac{A}{P} = e^{rt} \]

take natural log of both sides

\[
\ln\left(\frac{A}{P}\right) = \ln(e^{rt}) = rt
\]

divide by \( t \)

\[
\ln\left(\frac{A}{P}\right) = \frac{rt}{t} = r
\]

Solved for \( r \)
Example: Deposit $937 into account with 2.3% interest compounded continuously. How long until the balance has grown to $1200.

(Exact answer, then decimal approx)

Solution: $\text{P} = 937$

$\text{r} = 0.023$

$\text{A} = 1200$

$t = \text{unknown}$

Use the formula that gives us $t$. 

First
\[ t = \frac{\ln \left( \frac{A}{P} \right)}{r} = \frac{\ln \left( \frac{1200}{937} \right)}{0.023} \times 10.25 \text{ years} \]

exact answer

decimal approximation

end of lecture