Section 3.1

Today! Periodically-Compounded Interest

Start with “Simple Interest”

Idea

Bank account earns interest only on the money that was originally deposited.

Simple Interest Formula

\[ A = P + Prt = P(1 + rt) \]

- \( P \) = principal = money originally deposited
- \( r \) = interest rate expressed as a decimal
- \( t \) = time in years since original deposit
- \( A \) = amount of money in account at time \( t \).
Example
Deposit $1000 into account with 2% simple interest.

What would be the balance at times $t=0$, $t=1$, $t=5$?

Solution

At $t=0$  
$$A = 1000 \left(1 + 0.02(0)\right) = 1000(1) = $1000$$

At $t=1$  
$$A = 1000 \left(1 + 0.02(1)\right) = 1000 \left(1.02\right) = $1020$$

At $t=5$  
$$A = 1000 \left(1 + 0.02(5)\right) = 1000 \left(1 + 1\right) = 1000(2) = $2000$$
Plot this information (Plot Balance $A$ as a function of time $t$)

Balances are in dollars $A = P + Prt = (Pr)t + P$

$y$-intercept $(0, P) = (0, 1000)$

slope $m = Pr = 1000(0.02) = 20$

Point $(5, 1100)$

$t = 0$
$t = 1$
$t = 5$

time in years
Periodically Compounded Interest

New idea for what to do with our $1000
Four times per year, withdraw all money from account. Start a new account (earning simple interest at 2%) with all that money.
Do that for 5 years.

Figure out the balance resulting after five years.

<table>
<thead>
<tr>
<th>time</th>
<th>Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t = 0$</td>
<td>Balance $= A = 1000$</td>
</tr>
<tr>
<td>$t = \frac{1}{4}$ year</td>
<td>$A = 1000 \left(1 + 0.02 \left(\frac{1}{4}\right)\right) = 1005$</td>
</tr>
<tr>
<td>$t = \frac{1}{2}$ year</td>
<td>$A = 1000 \left(1 + 0.02 \left(\frac{1}{4}\right)\right) \cdot \left(1 + 0.02 \left(\frac{1}{4}\right)\right) = $1040.025</td>
</tr>
<tr>
<td>$t = \frac{3}{4}$ year</td>
<td>$A = 1000 \left(1 + 0.02 \left(\frac{1}{4}\right)\right) \cdot 2 \cdot \left(1 + 0.02 \left(\frac{1}{4}\right)\right) \approx $1017.25</td>
</tr>
<tr>
<td>$t = \frac{5}{4}$ year</td>
<td>$A = 1000 \left(1 + 0.02 \left(\frac{1}{4}\right)\right) \cdot 3 \cdot \left(1 + 0.02 \left(\frac{1}{4}\right)\right) \approx $1015.075</td>
</tr>
</tbody>
</table>
The final balance

\[ t = 5 \text{ years} \quad A = 1000 \cdot \left(1 + (0.02)(\frac{1}{4})\right)^{4 \cdot 5} \]

Notice: this is a bit more than the $1100 that we got from doing simple interest.
Graph the Balance of what we did

- Interest compounded \( m = \frac{1}{4} \) times per year
- \( (5, 1104.90) \)
- \( m = Pr = 2.0 \)
- Simple interest \( A = Pr + P \)

Points:
- \((0, 100)\)
- \((5, 1100)\)
Periodically-compounded Interest

\[ A = P \left(1 + \frac{r}{m}\right)^{mt} \]

- Formula only valid every \( \frac{1}{m} \) years
- \( P \) = initial deposit
- \( r \) = interest rate (expressed as decimal)
- \( m \) = number of times per year that the interest is "compounded"
- \( t \) = total time in years,
More examples with this formula

Example #3
Deposit $1000 into account with interest 2% compounded weekly \((n = 52)\). What will be the balance after 5 years?

\[
A = 1000 \left(1 + \frac{0.02}{52}\right)^{52 \times 5} \approx \$1105.15
\]

Solution

It seems like compounding more frequently leads to a bigger balance after 5 years.

Example #4
$1000 deposit, 2\%$ interest, 5 years, compounded daily \(m = 365\)

Find resulting balance.

Solution:
\[
A = 1000 \left(1 + \frac{0.02}{365}\right)^{365 \times 5} \approx \$1105.12
\]
Observe

More frequent compounding $\Rightarrow$ resulting balance $A$ bigger, but the balance $A$ seems to be levelling off.

Abbreviate this more:

as $m \to \infty$ it seems like $A$ increases, but $A$ seems to approach some limit.

Obvious question: Is there a limit?

That is what is $\lim_{m \to \infty} A$?
Question:

What is \( \lim_{{m \to \infty}} \frac{P(1 + \frac{r}{m})^m}{{n}^n} \) ??

Related simpler question:
What is the \( \lim_{{n \to \infty}} \left( 1 + \frac{1}{n} \right)^n \) ??

- Base getting closer and closer to 1
- Being raised to higher and higher powers

Which will win?

Will the limit be 1 or \( \infty \) ??
\[ \begin{array}{l|l}
\hline
n & \text{value of } (1 + \frac{1}{n})^n \\
\hline
10 & (1 + \frac{1}{10})^{10} \approx 2.5937 \\
100 & (1 + \frac{1}{100})^{100} \approx 2.70481 \\
1000 & (1 + \frac{1}{1000})^{1000} \approx 2.71692 \\
10,000 & \approx 2.71815 \\
\hline
\end{array} \]

It seems like \( \lim_{n \to \infty} (1 + \frac{1}{n})^n \) exists and is a number near 2.718.

Is this true?

Is there really a limit?

What is the value of the limit?