Continuing Section 2.4: The Derivative

Yesterday, given a graph of a function $f(x)$, we built a graph for $f'(x)$ by drawing tangent lines on $f(x)$ and finding their slopes.

Today: Analytical Examples. Given formula for $f(x)$

Goal: Calculate formula for $f'(x)$ using The Definition of the Derivative

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]
Example #1 \( f(x) = x^2 - 2x - 3 \)

Find \( f'(x) \) using the definition of the derivative.

Solution

\[
  f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

Cannot plug in \( h=0 \) because it leads to division by 0.
(If it leads to \( \frac{0}{0} \))

\[
  = \lim_{h \to 0} \frac{(x^2 + 2xh + h^2 - 2x + 2h - 3) - (x^2 - 2x - 3)}{h}
\]

Still cannot plug in \( h=0 \), it would lead to \( \frac{0}{0} \)

\[
  = \lim_{h \to 0} \frac{2xh + h^2 - 2h}{h}
\]
Details

\[ f(\alpha) = x^2 - 2x - 3 \]

\[ f(x) = (\quad)^2 - 2(\quad) - 3 \quad \text{empty version} \]

\[ f(x+h) = (x+h)^2 - 2(x+h) - 3 \]

\[ = x^2 + 2xh + h^2 - 2x - 2h - 3 \]

\[ (x+h)^2 = (x+h)(x+h) \]

\[ = x^2 + 2xh + h^2 \]

\[ = x^2 + 2xh + h^2 \]
\[ \lim_{h \to 0} \frac{h(2x + h - 2)}{h} \]

Factored out an \( h \) in numerator

Since \( h \to 0 \), we know \( h \to 0 \)
So we can cancel \( \frac{h}{h} \)

\[ \lim_{h \to 0} 2x + h - 2 \]

Polynomial in the variable \( h \)
So we can use theorem 3
And just substitute in \( h = 0 \)

\[ = 2x + (0) - 2 \]

\[ = 2x - 2 \]

Conclusion: For \( f(x) = x^2 - 2x - 3 \)
We found \( f'(x) = 2x - 2 \)
Discuss: how would the graphs of $f(x)$ and $f'(x)$ look?

$f(x) = x^2 - 2x - 3 = (x+1)(x-3)$

- parabola
- facing up
- $y$-intercept at $(x, y) = (0, -3)$
- $x$-intercepts at $(x, y) = (-1, 0)$ and $(3, 0)$

$f'(x) = 2x - 2$

- line
- slope: $m = 2$
- $y$-intercept: $(x, y) = (0, -2)$

This is an analytical version of the Graphical Example that we did yesterday in Class Drill 6.
The rest of today: more analytical examples involving Definition of Derivative

Polynomial Example

\( f(x) = -3x^2 + 5x - 7 \)

Find \( f'(x) \) using the Definition of the Derivative.

(Very similar to previous example.
The only complication is the -3 coefficient.
You study this one at home.)

Advice: Build your solution by mimicking my example above + book examples.
In particular: Identify the "empty version" of \( f \).
Example \( f(x) = \frac{1}{x} \) find \( f'(x) \) using definition of derivative.

\[
\frac{f'(x)}{f(x)} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\left( \frac{1}{x+h} \right) - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{1}{h} \left( \frac{1}{x+h} - \frac{1}{x} \right)
\]

\[
= \lim_{h \to 0} \frac{1}{h} \left( \frac{x - (x+h)}{x(x+h)} \right) = \lim_{h \to 0} \frac{1}{h} \left( \frac{x - x - h}{x(x+h)} \right)
\]

\[
= \lim_{h \to 0} \frac{1}{h} \left( \frac{-h}{x(x+h)} \right) = \frac{1}{x^2} \text{ empty version}
\]

\[
f'(x) = \frac{1}{x^2}
\]
\[
\lim_{{h \to 0}} \frac{1}{h} \left( \frac{x - (x+h)}{x(x+h)} \right)
\]

\[
= \lim_{{h \to 0}} \frac{1}{h} \cdot \left( \frac{-h}{x^2 + xh} \right)
\]

Since \( h \to 0 \), we know \( h \neq 0 \)
so we can cancel \( \frac{h}{h} \)

\[
= \lim_{{h \to 0}} \frac{1}{x^2 + xh}
\]

Rational function in variable \( h \)
and \( h = 0 \) is in domain
so we can substitute \( h = 0 \)

\[
= -\frac{1}{x^2 + x(0)}
\]

\[
= -\frac{1}{x^2}
\]
Conclusion: For \( f(x) = \frac{1}{x} \) we find \( f''(x) = -\frac{1}{x^2} \)

Harder Example: For \( f(x) = 5 - \frac{17}{x} \), find \( f''(x) \) using definition of derivative.

\[
\frac{f'(x)}{f''(x)} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{h} \left( f(x+h) - f(x) \right)
\]

\[
= \lim_{h \to 0} \frac{1}{h} \left( \frac{5}{x+h} - \frac{5}{x} \right)
\]

\[
= \lim_{h \to 0} \frac{1}{h} \left( \frac{5}{x+h} - \frac{5}{x} \right)
\]

\[= \lim_{h \to 0} \frac{1}{h} \left( \frac{-17}{x} - \frac{-17}{x+h} \right)\]
It is useful to pull a constant out front. I'll do that in stages

\[
= \lim_{h \to 0} \frac{1}{h} \left( \frac{-17}{x+h} - \frac{-17}{x} \right)
\]

Showed that each term contains a multiplicative factor of \((-17)\)

Factored out the \((-17)\)

\[
= \lim_{h \to 0} \frac{1}{h} \left( -17 \left( \frac{1}{x+h} - \frac{1}{x} \right) \right)
\]

Since multiplication is commutative, we can move the \((-17)\) past the \(\frac{1}{h}\)

\[
(-17) \cdot \lim_{h \to 0} \left( \frac{1}{h} \cdot \left( \frac{1}{x+h} - \frac{1}{x} \right) \right)
\]

Limit Theorem 2.5 from section 2.1 tells us that \(\lim (K \cdot f(x)) = K \cdot \lim (f(x))\)

That is, a multiplicative constant can be pulled right past a limit symbol

\[
= (-17) \cdot \left( \frac{-1}{x^2} \right)
\]

\[
= \frac{17}{x^2}
\]
Conclusion:

We have found that

If \( f(x) = 5 - \frac{17}{x} \)

then \( f'(x) = \frac{17}{x^2} \)