Day 3 (Wed Jan 13, 2016)

Remember:
- Sign in
- Quiz 1 on Friday
- Buy your course packers
- Put your phones away.

Today: Start with leftovers from Section 2.1

Limits of Difference Quotients

Example similar to 2.1 #67

For \( f(x) = x^2 - 6x + 5 \),

\[
\lim_{{h \to 0}} \frac{f(4+h) - f(4)}{h} \quad \text{(the limit of a difference quotient)}
\]

\[
\text{a “difference quotient”}
\]
Solution

First notice that if we substitute in $h=0$, we get

$$\frac{f(4+h) - f(4)}{h} = \frac{f(4) - f(4)}{0} = \frac{0}{0}$$

But this does not tell us that the limit is undefined. It only tells us that we have to use a different method.

We will use "algebraic simplification" as we did yesterday.
\[ f(4) = (4)^2 - 6(4) + 5 \]
\[ f(4+h) = (4+h)^2 - 6(4+h) + 5 \]
\[ = (4+h)(4+h) - 6(4) - 6(h) + 5 \]
\[ = 4^2 + 4h + 4h + h^2 - 6(4) - 6(h) + 5 \]

Compute the limit:
\[ \lim_{h \to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h \to 0} \frac{(4^2 + 8h + h^2 - 6(4) - 6h + 5) - (4^2 - 6(4) + 5)}{h} \]
\[ = \lim_{h \to 0} \frac{8h + h^2 - 6h}{h} \]
\[ = \lim_{h \to 0} \frac{2h + h^2}{h} \]
\[ = \lim_{h \to 0} \frac{h(2+h)}{h} \]
\[ = \lim_{h \to 0} 2 + h \]
\[ = 2 \]

Since \( h \to 0 \), we know \( h \neq 0 \) so we can cancel \( \frac{h}{h} \)

Now we are taking limit of a polynomial, so the 3 says we can substitute \( h \to 0 \)
Conclusion: \( \lim_{{n \to 0}} \frac{f(x+n) - f(x)}{n} = 2 \)

Another example involving the limit of a rational function:

\[ f(x) = \frac{(x+13)}{(x^2+13x)} \quad \frac{x^2 - 6x + 5}{x^2 - 8x + 15} = \frac{(x-1)(x-5)}{(x-3)(x-5)} \]

Find \( \lim_{{x \to 5}} f(x) = \lim_{{x \to 5}} \frac{(x-1)(x-5)}{(x-3)(x-5)} \)

Since \( x \to 5 \) we know \( x \neq 5 \)

So \( x - 5 \neq 0 \)

So we can cancel \( \frac{x-5}{x-5} \)

This is the limit of a rational function, and \( x=5 \) is in the domain.

So theorem 3 tells us that we can just substitute in \( x=5 \)

\[
\begin{align*}
\lim_{{x \to 5}} f(x) &= \frac{5-1}{5-3} \\
&= \frac{4}{2} \\
&= 2
\end{align*}
\]
Now for the same function $f(x)$, find $\lim_{x \to 3} f(x)$.

Solution:

$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{(x-1)(x-5)}{(x-3)(x-5)}$

$= \lim_{x \to 3} \frac{x-1}{x-3}$

Since $x \to 3$, we know that $x$ is close to 3, so $x \neq 3$ and also $x \neq 5$ either. So we can cancel the $\frac{x-5}{x-5}$.

Notice: $x$ is getting closer and closer to 3.

The limit of the numerator is

$\lim_{x \to 3} x-1 = 3-1 = 2$

$\uparrow$ Theorem 3

The limit of the denominator is

$\lim_{x \to 3} x-3 = 3-3 = 0$

$\uparrow$ Theorem 3

Since the limit of the numerator is 2 and the limit of the denominator is 0, Theorem 4 tells us that the limit does not exist!
Let's explore what's going on with the function

\[ f(x) = \frac{x^2 - 6x + 15}{x^2 - 8x + 15} = \frac{(x-1)(x-5)}{(x-3)(x-5)} \]

We'll do that in Class Drill 3
Class Drill 3: Guessing Limits by Substituting in Numbers

Without using a calculator, answer the following questions about the function

\[ f(x) = \frac{x^2 - 6x + 5}{x^2 - 8x + 15} \]

Part 1: Function Values

(1) Factor \( f \). (Check your factorizations by multiplying.)

\[ f(x) = \frac{(x-1)(x-5)}{(x-3)(x-5)} \]

\[ (x-1)(x-5) = x^2 - x - 5x + 5 = x^2 - 6x + 5 \]

\[ (x-3)(x-5) = x^2 - 3x - 5x + 15 = x^2 - 8x + 15 \]

(2) Are you allowed to cancel factors in the factored form of \( f \)? Explain why you think you are allowed to cancel, or why you are not.

No, can't cancel because that would change the function into a different function (with a different domain).

(3) Find \( f(1) \) by substituting \( x = 1 \) into the factored version of \( f \).

\[ f(1) = \frac{(1-1)(1-5)}{(1-3)(1-5)} = \frac{0(-4)}{-2(-4)} = \frac{0}{-2} = 0 \]

(4) Find \( f(3) \) by substituting \( x = 3 \) into the factored version of \( f \).

\[ f(3) = \frac{(3-1)(3-5)}{(3-3)(3-5)} = \frac{2(-2)}{0(-2)} = \frac{2}{0} \text{ undefined!} \]

(5) Find \( f(5) \) by substituting \( x = 5 \) into the factored version of \( f \).

\[ f(5) = \frac{(5-1)(5-5)}{(5-3)(5-5)} = \frac{4(0)}{2(0)} = \frac{0}{0} \text{ undefined} \]

Part 2: Limits

Using the factored form of \( f \), compute the following values and guess the limits.

Guessing the limit at \( x = 5 \).

(Just leave answers as an expression ready to type into a calculator.)

(11) \( f(5.1) = \frac{(5.1-1)(5.1-5)}{(5.1-3)(5.1-5)} = \frac{4.1}{3.1} \)

(12) \( f(5.01) = \frac{(5.01-1)(5.01-5)}{(5.01-3)(5.01-5)} = \frac{4.01}{2.01} \)

(13) \( f(5.001) = \frac{(5.001-1)(5.001-5)}{(5.001-3)(5.001-5)} = \frac{4.001}{2.001} \)

(15) \( \lim_{x \to 5^+} f(x) = 2 \) this agrees with the example that we did on page 4.
(16) \[ f(4.9) = \frac{(4.9 - 1)(4.9 - 5)}{(4.9 - 3)(4.9 - 5)} = \frac{3.9}{1.9} \]

(17) \[ f(4.99) = \frac{3.99}{1.99} \]

(18) \[ f(4.999) = \frac{3.999}{1.999} \]

(20) Guess \( \lim_{x \to 5} f(x) = 2 \)

(21) Guess \( \lim_{x \to 5} f(x) = 2 \) this agrees with our result from example on page 41

Guessing the limit at \( x = 3 \). (Simplify your answers.)

(11) \[ f(3.1) = \frac{(3.1 - 1)(3.1 - 5)}{(3.1 - 3)(3.1 - 5)} = \frac{2.1}{.1} = 21 \]

(12) \[ f(3.01) = \frac{(3.01 - 1)(3.01 - 5)}{(3.01 - 3)(3.01 - 5)} = \frac{2.01}{.01} = 201 \]

(13) \[ f(3.001) = \frac{2.001}{.001} = 2001 \]

(15) Guess \( \lim_{x \to 3^+} f(x) = \infty \) (See remarks on page 9)

(16) \[ f(2.9) = \]

(17) \[ f(2.99) = \]

(18) \[ f(2.999) = \]

(20) Guess \( \lim_{x \to 3^-} f(x) = \)

(21) Guess \( \lim_{x \to 3} f(x) = \)
Observe

As \( x \) gets closer \(+\) close to 3 \( +\) from the right, the \( y \)-values get more \( +\) more positive without bound.

In section 2.1 we would have said that
\[
\lim_{{x \to 3^+}} f(x) \text{ DNE}
\]

But now in section 2.2 we will introduce a new terminology
\[
\lim_{{x \to 3^+}} f(x) = \infty
\]

The limit is infinity!