Exam 2 moved to Wednesday, Oct 14. I will update the course schedule on the webpage.

Continuing discussion of Ch. 6 Angle Measurement

Axioms \langle N7 \rangle, \langle N8 \rangle, \langle N9 \rangle are about the angle measurement function \( M: A \rightarrow (0, 180) \)

Obvious Questions
Is the function \( M \) one-to-one?

No, for example consider \( \angle ABC \) and \( \angle DEF \)

\[
\begin{align*}
\angle ABC &= 37 \\
\angle DEF &= 37 \\
\end{align*}
\]

Even though \( \angle ABC \neq \angle DEF \)
Is the function m onto?

Intuitively, yes. Given number r with \(0 < r < 180\), there is some angle \(\angle{LABC}\) such that

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Section 6.3

Th. 39 About points in interior of angles

Given: Points C, D on same side of line \(\overline{AB}\)

Claim: The following are equivalent:

1. D is in interior of \(\angle{ABC}\)
2. \(m(\angle{ABD}) < m(\angle{ABC})\)

Note that this theorem does not say that either statement is true. It only says that they are either both true or they are both false.
Example #1

(1) is true
(2) is true

Example #2

(1) is false
(2) is false

Proof is not above the level of this course, but we won't study it, because the proof style is similar to another proof that we will study later.

Definition 43 of Angle Bisector

See Book

Class Drill: Justify & Illustrate steps in proof of Th. 40 Every Angle has a unique bisector.
Justify the steps in the proof of the following theorem. Draw a picture to illustrate. (Your justifications may refer to any prior theorem and to Neutral Axioms <N1> through <N9>. You may not use Axiom <N10>.)

**Theorem 40:** Every angle has a unique bisector.

**Proof**

1. Suppose that angle $\angle ABC$ is given. **(Make a drawing.)**

   ![Diagram of angle ABC]

2. Introduce special ray $\overline{BD}$ and show that it is a bisector of $\angle ABC$. **(Existence)**
   
   - The angle $\angle ABC$ has a measure that is a real number denote by the symbol $m(\angle ABC)$.
     
     (Justify)
     
     by axiom $\langle N7 \rangle$ there is an angle measurement function.

3. Let $r = \frac{1}{2} m(\angle ABC)$. Let $H_C$ be the half-plane created by line $\overline{AB}$ that contains point $C$.
   
   Observe that ray $\overline{BA}$ lies on the edge of this half-plane. **(Make a new drawing.)**

   ![Diagram of half-plane H_C]

4. There exists a ray $\overline{BD}$ such that $D \in H_C$ and $m(\angle ABD) = r$. **(Justify and make a new drawing.)**

   By angle construction axiom $\langle N8 \rangle$ existence part
   
   applied ray: $\overline{BA}$
   
   half plane: $H_C$
   
   number: $r$

   Axiom $\langle N8 \rangle$ says that ray $\overline{BD}$ exists.
(5) Point \( D \) is in the interior of \( \angle ABC \). (by statements (3), (4), and Theorem 39) (Make a new drawing.)

From statements (3), (6) we have the following:
- \( C, D \) are in the same half plane.
- Ray \( BA \) is on the edge of that half plane.
- \( m(\angle ABD) < m(\angle ABC) \)

Theorem 39 II \( \Rightarrow \) I tells us that \( D \) is in the interior.

(6) \( m(\angle ABD) = m(\angle DBC) \). (Justify) (This will take 2 or 3 steps)

we know \( D \) is interior of \( \angle ABC \)
by (5).

So therefore \( m(\angle ABC) = m(\angle ABD) + m(\angle DBC) \)
by \( m(\angle ABC) \) Angle measure addition.

\[
\frac{1}{2} m(\angle ABC) = \frac{1}{2} m(\angle ABD) + \frac{1}{2} m(\angle DBC) \]  

(Substituted in fact that \( m(\angle ABD) = \gamma = \frac{1}{2} m(\angle AAB) \))

\[
\frac{1}{2} m(\angle ABC) = m(\angle DBC) \]  

(Subtracted \( \frac{1}{2} m(\angle ABC) \))

(7) Ray \( \overrightarrow{BD} \) is a bisector of \( \angle ABC \). (Justify) (Make a new drawing.)

Observe:
- Endpoint of \( \overrightarrow{BD} \) is point \( B \) which is vertex of \( \angle ABC \)
- \( D \) is in interior
- we have proven that \( m(\angle ABD) = m(\angle DBC) \)

So, Ray \( \overrightarrow{BD} \) fulfills all requirements of Definition 43 of angle bisector.
Show that ray $BD$ is the only bisector of $\angle ABC$. 

(8) Suppose that ray $BD'$ is a bisector of $\angle ABC$. (Make a new drawing.)

\[ \begin{array}{c}
\text{C} \\
\text{B} \\
\text{A} \\
\text{D}'
\end{array} \]

(9) Point $D'$ is in the interior of $\angle ABC$ and $m(\angle ABD') = m(\angle D'BC)$. (Justify and make a new drawing.)

Definition #43 of bisector tells us that all this stuff is true.

(10) $m(\angle ABD') = \frac{1}{2} m(\angle ABC)$. (Justify)

\[ m(\angle ABC) = m(\angle ABD') + m(\angle D'BC) \]  
by angle addition, applied point D'

which is known to be in interior of $\angle ABC$

= $m(\angle ABD') + m(\angle ABD')$  
Because of step 9

= 2 $m(\angle ABD')$

Divide by 2

\[ \frac{1}{2} m(\angle ABD) = m(\angle ABD') \]
(11) Points $D'$ and $C$ are on the same side of line $AB$. (Justify and make a new drawing.)

We know that $D'$ is in interior of $\angle ABC$ (from statement 9).

Def. of angle interior tells us that $C, D'$ must be on same side of $\overline{AB}$.

(12) Point $D'$ is in half-plane $H_C$. (Justify and make a new drawing.)

Just rephrasing (11).

(13) Ray $\overline{BD'}$ is the same ray as $\overline{BD}$. (Justify and make a new drawing.)

We know that
- $D'$ is in half-plane $H_C$
- $m(\angle ABD') = \gamma$

Axiom $\angle N8$ uniqueness part tells us that there is only one ray that has those properties.

End of Proof