Chapter 2  Axiomatic Geometries

Section 2.1  What is an "axiomatic geometry"?

Section 2.1.5  Recurring Questions about Parallelism

Consider the answers for 5-point geometry:

Objects: Points, lines
Relation: Point lies on the line

Axioms:
1. There exist exactly 5 points
2. Given any two distinct points, there exists exactly one line that both points lie on.
3. Given any line, there exist exactly two points that lie on that line.
Model

5 dots
10 line segments

Question 1: Do parallel lines exist?

Answer: yes!

Example

What about these?

These are parallel
(no shared points)
Abstract proof that parallel lines exist

Proof

1) There exist 5 points (by axiom (1))
   we can call them P₁, P₂, P₃, P₄, P₅

2) There exists a line that P₁, P₂ lie on (by axiom (2))
   call it Lₐ

3) There exists a line that P₃, P₅ lie on (by axiom (2))
   It can't be line Lₐ because Lₐ is full (by axiom (2))
   So it has to be a new line. Call it L₉.

4) Note Lₐ can only only points P₁, P₂ lie on Lₐ
   using axiom (3)
   and only points P₃, P₅ lie on L₉ (using (3))
   So Lₐ and L₉ are parallel (by definition of parallel)

End of proof
What about the other recurring question? Given a line $L$ and a point $P$ not on $L$, how many lines exist that pass through $P$ and are parallel to $L$? Consider model for inspiration.

Given line $L$
Given point $P$ not on $L$

It looks like there are two lines that pass through $P$ and are parallel to $L$. 

\[4\]
Example

Given line $L$
Point $P$ not on $L$

How could the models tell us the answer is 2?
How do we prove it abstractly?

First, state the claim clearly.
Claim I. 5-point geometry

Given a line $L$ and a point $P$ not on $L$, there exist exactly two lines $M, N$ that pass through $P$ and are parallel to $L$.

Proof Structure

1. Suppose a line $L$ and a point $P$ not on $L$ are given.
   
5. We have shown that there exist exactly two lines $M, N$ that pass through $P$ and are parallel to $L$. 
Fill in steps

(1) Given line $L$ and $P$ not on $L$

(2) There exist points $P_1, P_2$ on $L$
   by axiom $\Theta$

(3) There exist two more points $P_3, P_4$
   (by axiom $\Delta$)
   and they don't lie on $L$
   (by statement (2))

Now this gap in the proof is not so wide:
Fill in the missing steps.

(End) We have shown that there exist exactly two lines $M, N$ that
pass through $P$ and are parallel to $L$. 

[end of lecture]