Section 1.2 Properties of Axiom Systems

Consistency

An axiom system is said to be consistent if there exists a model for the axiom system. (An interpretation that is successful.)

Example

The 5-axle axiom system (from Monday) is consistent. (Proof: we have Bob's model.)
What would be an axiom system that is not consistent?

It would be an axiom system for which no model exists. An axiom system with contradiction in its axioms.

Example: Dave's Axiom System

- Primitive terms: ake, bem
- Primitive relation: the ake is related to the bem.

Axioms:

1. Axioms same as the axioms for the 5-ake axiom system from Monday
2. There is exactly one bem.

Using axioms 1, 2, and 3 we can prove that there are exactly 10 bems. This contradicts axiom 4.
Which is the "bad" axiom? We might think of <4> as the bad one, because it messed up an otherwise good axiom system.

But consider Ed's axiom system

<1> There are exactly 5 axioms
<2> For any two axioms, there is exactly one bwm that both are related to
<4> There is exactly one bwm.

Ed's axiom system is consistent. Here is a model.

-----

Moral: The problem with Dave's axiom system is that it can't be pinned on just axiom <4>. The problem is the whole set.
Independence

Dependent + Independent Axioms

Given a list of axioms, an axiom on the list is said to be dependent if it is possible to prove that that axiom must be true, using the other axioms in the proof.

(or if it is possible to prove that it must be false)
Example Consider Fred's List of Axioms

\(<1>\) \quad \{ \text{the three axioms from the 5-ake} \}
\(<2>\) \quad \text{axiom system from monday.} \)
\(<3>\) \quad \text{There are exactly 10 bems.} \)
\(<4>\) \quad \text{Axiom } \langle 4 \rangle \text{ is dependent. We can use Axioms} \langle 1 \rangle, \langle 2 \rangle, \langle 3 \rangle \text{ to prove there must be 10 bems.} \)

Now go back to Dave's Axiom System from earlier today

\(<1>\) \quad \{ 3 \text{ axioms from 5-ake axiom system.} \)
\(<2>\) \quad \{ \}
\(<3>\) \quad \{ \}
\(<4>\) \quad \{ \text{There is exactly 1 bem.} \}

We can use Axioms \langle 1 \rangle, \langle 2 \rangle, \langle 3 \rangle \text{ to prove that } \langle 4 \rangle \text{ is false so } \langle 4 \rangle \text{ is dependent.} \)
An axiom is said to be independent if it is impossible to prove that it is true or false based on the other axioms.

To show that an axiom is independent, one must come up with examples of interpretations where all other axioms are true and the given axiom is false, and all other axioms are true and the given axiom is false.
Example For $\mathcal{A}$ an axiomatic system
prove that axiom $\langle 1 \rangle$ is independent.

$\langle 1 \rangle \; T$
$\langle 2 \rangle \; T$
$\langle 3 \rangle \; T$

---

$\langle 1 \rangle \; F$
$\langle 2 \rangle \; T$
$\langle 3 \rangle \; T$

---

So even knowing $\langle 2 \rangle, \langle 3 \rangle$ are both true, it is still possible for $\langle 1 \rangle$ to be true or false.
Prove that axiom 2 is independent.

\[ \langle 1 \rangle T \]
\[ \langle 2 \rangle T \]
\[ \langle 3 \rangle T \]

So axiom \( \langle 2 \rangle \) can be either \( T \) or \( F \)
when \( \langle 1 \rangle, \langle 3 \rangle \) are both true.

---

Prove that axiom \( \langle 3 \rangle \) is independent.

\[ \langle 1 \rangle T \]
\[ \langle 2 \rangle T \]
\[ \langle 3 \rangle F \]

Since all three axioms are independent, we say that the set of axioms is independent. This is the definition of an Independent Axiom System.