Class Drill 5: Drill for Section 5.2 Justifying, Illustrating Steps in proof of Theorems 27, 28

Theorem 28 (Pasch’s Theorem) about a line intersecting a side of a triangle between vertices

If a line intersects the side of a triangle at a point between vertices, then the line also intersects the triangle at another point that lies on at least one of the other two sides.

Proof

(1) Suppose that line \( L \) intersects side \( \overline{AB} \) of \( \triangle ABC \) at a point \( D \) such that \( A \neq D \neq B \).
(2) Points \( A \) and \( B \) are on opposite sides of line \( L \). (Justify.)

Let \( H_A \) and \( H_B \) be their respective half-planes.
(3) Exactly one of the following statements is true. (Justify.)

(i) \( C \) lies on \( L \). (Make a drawing for case (i).)

(ii) \( C \) is in \( H_A \). (Make a drawing for case (ii).)

(iii) \( C \) is in \( H_B \). (Make a drawing for case (iii).)

Case (i)
(4) If \( C \) lies on \( L \), then \( L \) intersects both \( \overline{AC} \) and \( \overline{BC} \) at point \( C \). (Justify.)

Case (ii)
(5) If \( C \) is in \( H_A \), then points \( B \) and \( C \) lie on opposite sides of \( L \). (Justify.)

(6) In this case, \( L \) will intersect \( \overline{BC} \) at a point between \( B \) and \( C \). (Justify.)

Case (iii)
(7) If \( C \) is in \( H_B \), then points \( A \) and \( C \) lie on opposite sides of \( L \). (Justify.)

(8) In this case, \( L \) will intersect \( \overline{AC} \) at a point between \( A \) and \( C \). (Justify.)
Conclusion of cases

(9) In every case, we see that $L$ intersects $\overline{AC}$ or $\overline{BC}$ or both.

End of Proof

Theorem 29 about a line intersecting two sides of a triangle between vertices

If a line intersects two sides of a triangle at points that are not vertices, then the line cannot intersect the third side.

Proof

(1) Suppose that line $L$ intersects side of $\triangle ABC$ at a point $D$ on side $\overline{AB}$ and $E$ on side $\overline{AC}$, where neither $D$ nor $E$ is a vertex. (Make a drawing.)

(2) Points $A$ and $B$ are on opposite sides of line $L$. (Justify.) Let $H_A$ and $H_B$ be their respective half-planes.

(3) Points $A$ and $C$ are on opposite sides of line $L$. (Justify.) Therefore, $C$ is an element of half-plane $H_B$.

(4) Therefore, line $L$ does not intersect $\overline{BC}$. (Justify.)

End of Proof