Class Drill 2: Drill for Section 2.3: Theorems of Incidence Geometry

<table>
<thead>
<tr>
<th>Axiom System:</th>
<th>Incidence Geometry (Introduced in Section 2.3.2)</th>
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<tr>
<td>Primitive Objects:</td>
<td><em>point</em>, <em>line</em></td>
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<tr>
<td>Primitive Relations:</td>
<td><em>the point lies on the line</em></td>
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| Axioms: | `<I1>` There exist two distinct points.  
`<I2>` For every pair of distinct points, there exists exactly one line that both points lie on.  
`<I3>` For every line, there exists a point that does not lie on the line.  
`<I4>` For every line, there exist two points that do lie on the line. |

In this class drill, you will study the proofs of some of the Incidence Geometry theorems. There are four parts to the class drill. They are labeled [1], [2], [3], [4].

[1] Justify the steps in the following proof of Incidence Geometry Theorem 1.

Theorem 1: In Incidence Geometry, if *L* and *M* are distinct lines that intersect, then they intersect in only one point.

Proof
(1) In Incidence Geometry, suppose that *L* and *M* are distinct lines that intersect. (Justify.)

(2) Since lines *L* and *M* intersect, there must be at least one point that both lines *L* and *M* pass through (by definition of intersect). We can call one such point *P*.

(3) Assume that there is more than one point that that both lines pass through. (Justify.)

Then there is a second point, that we can call *Q*.
(4) Observe that there are two lines that pass through points *P* and *Q*.
(5) We have reached a contradiction. (Identify the contradiction.)

So our assumption in (3) was wrong. There cannot be more than one point that that both lines pass through.
(6) Conclude that lines *L* and *M* only intersect in one point. (by (1), (2), and (5))

End of Proof
[2] Justify the steps in the following proof of Incidence Geometry Theorem 2.

Incidence Geometry Theorem #2: In Incidence Geometry, there exist three points that are not collinear.

Proof

(1) In Incidence Geometry, two distinct points exist. (Justify.)

Call them P and Q.

(2) A line exists that passes through P and Q. (Justify.)

Call the line L.

(3) There exists a point that does not lie on L. (Justify.)

Call the point R.

(4) We already know (by statement (3)) line L does not pass through all three points P,Q,R. But suppose that some other line M does pass through all three points. (assumption)

(5) Observe that points P and Q both lie on line L and also both lie on line M.

(6) Statement (5) contradicts something. (Explain the contradiction.)

Conclude that our assumption in step (4) was wrong. That is, there cannot be a line M that passes through all three points P,Q,R. Conclude that the points P,Q,R are non-collinear.

End of Proof
Incidence Geometry Theorem #3: In Incidence Geometry, there exist three lines that are not concurrent.

**Proof:**

Part I: Introduce three lines L, M, N.

1. There exist three non-collinear points. (Justify.)

Call them A, B, C.

2. There exists a unique line that passes through points A and B. (Justify.)

Call it L.

3. Line L does not pass through point C. (Justify.)

4. Similarly, there exists a line M that passes through B and C and does not pass through A, and a line N that passes through C and A and does not pass through B.

Part II: Show that lines L, M, N are not concurrent.

5. Suppose that lines L, M, N are concurrent. That is, suppose that there exists a point that all three lines L, M, N pass through. (Justify.)

6. Any point that all three lines L, M, N pass through cannot be point A, B, or C. (Justify.)

So the point that all three lines pass through must be a new point that we can call point D.

7. There are two lines that pass through points A and D. (Justify.)

8. We have reached a contradiction. (Explain the contradiction.)

So our assumption in (5) was wrong. Lines L, M, N must be non-concurrent.

**End of Proof**
[4] Justify the steps in the following proof of Incidence Geometry Theorem 5.

Incidence Geometry Theorem #5: In Incidence Geometry, for every point P, there exist at least two lines that pass through P.

Proof
(1) In Incidence Geometry, suppose that a point P is given.
(2) There exist three non-collinear points. (Justify.)

(3) There are two possibilities:
   - Either P is one of the three points from statement (2)
   - or P is not one of the three points from statement (2)

Case 1
(4) Suppose that P is one of the three points from statement (2). Then let the other two points from statement (2) be named B and C. So the three non-collinear points are P,B,C.
(5) There exists a line through points P and B. (Justify.)
Call it $\overline{PB}$.
(6) There exists a line through points P and C. (Justify.)
Call it $\overline{PC}$.
(7) Lines $\overline{PB}$ and $\overline{PC}$ are not the same line. (Justify.)
(8) Therefore there are at least two distinct lines through P in this case

Case 2
(9) Suppose that P is not one of the three points from statement (2). Then let the three points from statement (2) be named A,B,C.
(10) Lines $\overline{PA}, \overline{PB}, \overline{PC}$ exist. (Justify.)
(11) Notice that is possible that two of the symbols in statement (10) could in fact represent the same line. For example, even though points P,A,B are distinct points, it could be that they are collinear. If that were the case, then the symbols $\overline{PA}, \overline{PB}$ would represent the same line. But regardless of whether or not P is collinear with certain pairs of the points A,B,C, we know that the three symbols $\overline{PA}, \overline{PB}, \overline{PC}$ cannot all three represent the same line. (Justify. That is, explain how we know that the symbols $\overline{PA}, \overline{PB}, \overline{PC}$ cannot all three represent the same line.)

That is, the three symbols $\overline{PA}, \overline{PB}, \overline{PC}$ represent at least two distinct lines, and possibly three.
(12) Therefore there are at least two distinct lines through P in this case

Conclusion
(13) Conclude that in either case, there exist at least two distinct lines through P.

End of Proof