Section 6.1 Area Between Curves

Remember the concept of signed and unsigned area.

The expression "Area between curves" refers to unsigned area. The book is not so clear about this.

Example: For the graph above, the "Area between graph of f and x-axis from x=a to x=b" mean the unsigned area, which is 14.
Theorem 1  About the area between two curves.

If \( \text{top}(x) \) and \( \text{bottom}(x) \) are continuous functions and \( \text{bottom}(x) \leq \text{top}(x) \) on the interval \([a, b]\)

Then the area bounded by the curves

\[
\begin{align*}
\text{top}(x) \\
\text{bottom}(x)
\end{align*}
\]

\( x = a \)
\( x = b \)

is

\[
\int_{a}^{b} \left( \text{top}(x) - \text{bottom}(x) \right) dx
\]
Example #1
Find area between graphs of $y = x^2 + 1$ and $y = 40$
from $x = -3$ to $x = 2$.

Solution: We have to figure out which function
is on the top.

We see that the "top"
graph is $y = 40$

$\text{USA} = \int_{x=-3}^{x=2} 40 - (x^2 + 1) \, dx$

$= \int_{x=-3}^{x=2} -x^2 + 39 \, dx$

$= \left[ -\frac{x^3}{3} + 39x + C \right]_{x=-3}^{x=2}$
\[
= \left( -\frac{(2)^3}{3} + 39(2) + c \right) - \left( -\frac{(-3)^3}{3} + 39(-3) + c \right)
\]
\[
= -\frac{8}{3} + 39(2 - (-3)) - \frac{27}{3}
\]
\[
= -\frac{8}{3} + 39(5) - \frac{27}{3}
\]
\[
= -\frac{8}{3} + 195 - 9
\]
\[
= -\frac{8}{3} + 186
\]
\[
= -\frac{8}{3} + \frac{558}{3}
\]
\[
= \frac{550}{3}
\]
Example #2  (In this example the top function changes)

Find the area bounded by the curves

\[ y = x^2 - 1 \]
\[ y = x - 1 \]
\[ x = -2 \]
\[ x = 1 \]

Solution: we have to figure out the top graph.
green region: \( x = -2 \) to \( x = 0 \), \( y = x^2 - 1 \) is on top
red region: \( x = 0 \) to \( x = 1 \), \( y = x - 1 \) is on top.

We will need two integrals

\[
\text{USA} = \text{Green Area} + \text{Red Area}
\]

\[
= \int_{-2}^{0} ((x^2 - 1) - (x - 1)) \, dx + \int_{0}^{1} ((x - 1) - (x^2 - 1)) \, dx
\]

Cancel before integrating

\[
= \int_{-2}^{0} x^2 - x \, dx + \int_{0}^{1} x - x^2 \, dx
\]

\[
= \left( \frac{x^3}{3} - \frac{x^2}{2} + C \right) \bigg|_{-2}^{0} + \left( \frac{x^2}{2} - \frac{x^3}{3} + K \right) \bigg|_{0}^{1}
\]
\[
\begin{align*}
&= \left(0^3 - 0^2 + 0\right) - \left((-2)^3 - (-2)^2 + c\right) + \left(\frac{1^2}{2} - \frac{1^3}{3} + k\right) - \left(\frac{0^2}{2} - \frac{0^3}{3} + k\right) \\
&= -\frac{-8}{3} + \frac{4}{2} + \frac{1}{2} - \frac{1}{3} \\
&= \frac{8}{3} + 2 + \frac{1}{2} - \frac{1}{3} \\
&= \frac{7}{3} + 2 + \frac{1}{2} \\
&= \frac{14}{6} + \frac{12}{6} + \frac{3}{6} \\
&= \frac{29}{6} = \text{USA} \\
\end{align*}
\]

Class Drill 29
The goal is to use two different methods find the (unsigned) area of the region bordered by the four lines:
- the line \( f(x) = 2x - 1 \)
- the line \( g(x) = -x + 2 \)
- the line \( x = -2 \)
- the line \( x = 3 \)

**Method #1: Finding the Unsigned Area Using Geometry**

Draw the four lines on the graph at right. Be sure to label the lines clearly.

Shade the region bordered by the four lines. (It should be two triangles.)

Find the *unsigned area* of the two triangles.
(Hint: Use the formula \( A = \frac{1}{2}bh \) for each triangle. The formula is easy to use if you choose the base to be the side of the triangle that is vertical.)

Write your results here:

Unsigned Area of Left Triangle = \( \frac{1}{2} (9)(3) = \frac{27}{2} \)

Unsigned Area of Right Triangle = \( \frac{1}{2} (6)(2) = 6 \)

Now add the two unsigned areas to find the total shaded area:

Total Shaded Area (unsigned area) =

\[
\text{USA} = \frac{27}{2} + 6 = \frac{27}{2} + \frac{12}{2} = \frac{39}{2}
\]
Method #2 Finding the Unsigned Area Using Calculus

Set up a sum of definite integrals to compute the *unsigned area*.
Your result should look like this.

\[
USA = \int_{x=a}^{x=b} \text{(some integrand here)} \, dx + \int_{x=b}^{x=c} \text{(another integrand here)} \, dx
\]

(You will have to figure out the integrands and the limits of integration \(a, b, c\).)
Then use calculus to find the value of the definite integrals and find their sum.

\[
USA = \int_{-2}^{1} (-x^2) - (2x-1) \, dx + \int_{1}^{3} (3x-2) - (-x^2) \, dx
\]

* Simplify before integrating!

\[
= \int_{-2}^{1} -3x + 3 \, dx + \int_{1}^{3} 3x - 3 \, dx
\]

\[
= \left[ \frac{-3x^2}{2} + 3x + C \right]_{-2}^{1} + \left[ \frac{3x^2}{2} - 3x + k \right]_{1}^{3}
\]

\[
= \left( \frac{-3(1)^2}{2} + 3(1) + C \right) - \left( \frac{-3(-2)^2}{2} + 3(-2) + C \right) + \left( \frac{3(3)^2}{2} - 3(3) + k \right) - \left( \frac{3(1)^2}{2} - 3(1) + k \right)
\]

\[
= \frac{-3}{2} + 3 + 6 + 6
\]

\[
= \frac{-3}{2} + 15
\]

\[
= \frac{27}{2}
\]

**Comparing The Numbers Obtained By The Two Methods**

Does the number that you obtained by Method #1 equal the number that you obtained by Method #2?

**Notice:** Method #1 gave \(USA = \frac{27}{2}\)

Method #2 also gave \(USA = \frac{39}{2}\)