Continuing Section 5.4 The Definite Integral
Consider an analytic example.
(Function f(x) given by formula, rather than a graph).

Let \( f(x) = 5 + \frac{x^2}{10} \)

Suppose we were interested in finding this signed area
\[
SA = \int_{x=2}^{x=12} f(x) \, dx = \int_{x=2}^{x=12} \left( 5 + \frac{x^2}{10} \right) \, dx
\]

On Tuesday, we tried to define this quantity as the Area of this region.
But this is not a simple geometric shape. So we don’t have an area formula.
But we can imagine an approximation of the area.

\[ S_A = \int_{x=2}^{x=12} f(x) \, dx = ? \]

Compute \( L_5 \): How wide are the rectangles?

\[
\text{Width} = \Delta x = \frac{\text{width of whole interval}}{\text{number of rectangles}} = \frac{b-a}{n} = \frac{12-2}{5} = \frac{10}{5} = 2
\]
How tall are the rectangles?

rectangle #1 touches graph at $x = 2$

height = $f(2) = 5 + \frac{2^2}{10} = 5 + \frac{4}{10}$

rectangle #2 touches graph at $x = 4$

height = $f(4) = 5 + \frac{(4)^2}{10} = 5 + \frac{16}{10}$

rectangle #5 touches graph at $x = 10$

height = $f(10) = 5 + \frac{10^2}{10}$
What is the Area of the rectangles?

Rectangle #1: \( A = \text{height} \cdot \text{width} = f(2) \cdot 2 = f(2) \cdot \Delta x \)

Rectangle #2: \( A = \text{height} \cdot \text{width} = f(4) \cdot \Delta x \)

Rectangle #3: \( f(6) \cdot \Delta x = A \)

Rectangle #4: \( A = f(8) \cdot \Delta x \)

Rectangle #5: \( A = f(10) \cdot \Delta x \)

Total area

\[ L_5 = f(2) \cdot \Delta x + f(4) \cdot \Delta x + f(6) \cdot \Delta x + f(8) \cdot \Delta x + f(10) \cdot \Delta x \]

\[ = (f(2) + f(4) + \cdots + f(10)) \cdot \Delta x = 94 \]

Similarly

\[ R_5 = (f(4) + f(6) + f(8) + f(10)) \cdot \Delta x = 122 \]
\[ L_5 = 94 \quad \text{<} \quad SA \quad \text{<} \quad R_5 = 122 \]

\[ L_{10} = 100.5 \quad \text{<} \quad SA \quad \text{<} \quad R_{10} = 114.5 \]

\[ L_{100} = 106.635 \quad \text{<} \quad SA \quad \text{<} \quad R_{100} = 108.035 \]

\[ L_{1000} = 107.263 \quad \text{<} \quad SA \quad \text{<} \quad R_{1000} = 107.403 \]
It seems like the values of $L_n$ are getting closer and closer to the values of $R_n$ as $n \to \infty$.

And they are both getting closer and closer to a common value near 10.73.

So it seems that the limit $\lim_{n \to \infty} L_n$ exists and the limit $\lim_{n \to \infty} R_n$ exists as well, and it seems like the limits have the same value:

$$\lim_{n \to \infty} L_n = \lim_{n \to \infty} R_n$$
Remember, we are studying approximations of the signed area \( SA = \int_{x=2}^{x=12} f(x) \, dx \) and we don't know how to define \( SA \) in this example.

\[ L_n \leq SA \leq R_n \]

Consider limit as \( n \to \infty \)

\[ \lim_{n \to \infty} L_n = SA = \lim_{n \to \infty} R_n \]

So, we will define the **Definite Integral** this way:

Definite integral of \( f \) from \( x=2 \) to \( x=12 \) is \( SA = \int_{x=2}^{x=12} f(x) \, dx = \lim_{n \to \infty} L_n = \lim_{n \to \infty} R_n \)