Discuss yesterday's example:

Find positive numbers $x, y$ such that

\[ 2x + y = 900 \]

\[ x, y = P \] is maximized

Solution structure:

- Eliminated variable $y$ to obtain function

\[ P(x) = 900 - 2x^2 = 2x(450 - x) \]

- Standard form: $P(x) = 900 - 2x^2$
- Factored form: $P(x) = 2x(450 - x)$

- Analyzed domain: $0 < x < 450$
- Used calculus to find critical number: $x = 225$

- Realizing that graph of $P(x)$ is a parabola facing down, we knew that $x = 225$ had to be the location of absolute max.
Alternate solution method: Steps labeled ⋆ could be done without calculus just by considering graph of \[ P(x) = 900x - 2x^2 = 2x(450-x) \]

Standard form tells us: parabola, facing down, vertex axis intercept \((x, P) = (0, 0)\)

Factored form tells us: horizontal axis intercepts at \((x, P) = (0, 0)\) and \((x, P) = (450, 0)\)

Clearly \(x = 225\) leads to max value of \(P(x)\) (don't need calculus)
The non-calculus solution is much simpler in this problem!

If we were given a choice of methods, the non-calculus method would be smarter.
New Examples

Example #1
Find positive numbers \( x, y \) such that
- the product is 9000
- the sum \( 10x + 25y \) is minimized

Solution

\[ \text{Step 1 Set up equations} \]
\[ \begin{align*}
3xy &= 9000 \quad \text{equation I} \\
10x + 25y &= S \quad \text{equation II}
\end{align*} \]

Need to minimize \( S \).

\[ \text{Step 2 Use Equation I to Eliminate } y \text{ in equation II} \]

Solve Equation I for \( y \) in terms of \( x \).
\[ x\, y = 9000 \]
\[ y = \frac{9000}{x} \]
Substitute into Equation II

\[ 10x + 25 \left( \frac{9000}{x} \right) = S \]

\[ 10x + \frac{25 \cdot 9000}{x} = S \]

This is an equation involving \( x \), \( S \) and it is solved for \( S \) in terms of \( x \).

So \( S \) is a function of the variable \( x \).

\[ S = 10x + \frac{25 \cdot 9000}{x} \quad \text{equation form} \]

\[ S(x) = 10x + \frac{25 \cdot 9000}{x} \quad \text{function form} \]

What is the domain? We know that

We are given that \( x \) must be positive. So \( x > 0 \) is the domain.
Step 3: Find the value of $x$ that minimizes $S(x)$.

The form of $S(x)$ is unfamiliar, so we can't use simple geometric argument.
We will need to use calculus.
Maxs & mins can only occur at critical numbers & endpoints.
We have no endpoints in this problem.
So in this problem, maxs & mins can only occur at critical numbers.

Find critical numbers for $S(x)$.

Start by looking for partition numbers for $S'(x)$
x-values such that $S'(x)$ DNE or $S'(x) = 0$. 
Looking for partition numbers for $S'$

So we need derivative of $S(x) = 10x + \frac{25.9000}{x}$

We could use quotient rule but smarter approach would be to first rewrite $S(x)$

Rewrite $S(x) = 10x + 25.9000 \frac{1}{x} = 10x + 25.9000x^{-1}$

Now find $S'(x)$

$S'(x) = \frac{d}{dx}(10x + 25.9000x^{-1})$

$= 10 \frac{dx}{dx} + 25.9000 \frac{dx}{dx}$

$= 10(1) + 25.9000(1)x^{-1}$

$= 10 - 25.9000x^{-2}$

$= 10 - \frac{25.9000}{x^2}$
Are there any $x$-values that cause $S'(x)$ to not exist?

Yet: $x = 0$ but this number is not.

Are there any $x$-values that cause $S'(x) = 0$?

Check $0 = 10 - \frac{25.9000}{x^2}$

\[
\frac{25.9000}{x^2} = 10
\]

\[
25.9000 = 10x^2
\]

\[
\frac{25.9000}{10} = x^2
\]

\[
25.900 = x^2
\]

$x = \sqrt{25.900}$ or $x = -\sqrt{25.900}$

\[
= 5.30 \text{ or } -5.30
\]

\[
= 150 \text{ or } -150
\]
So partition numbers for \( S'(x) \) are:

- \( x = -150 \) \( \iff \) \( S' = 0 \)
- \( x = 0 \) \( \iff \) \( S' \) to not exist
- \( x = 150 \) \( \iff \) \( S' = 0 \)

Which of these are also critical numbers for \( S(x) \)?

See if \( S(x) \) exists:

- \( S(-150) = 10(-150) + \frac{25 - 900}{-150} \) exists \( \checkmark \)
- \( S(0) = 10(0) + \frac{25 - 9000}{0} \) DNE !!
- \( S(150) = 10(150) + \frac{25 - 9000}{150} \) exists \( \checkmark \)

Only critical numbers for \( S(x) \) are \( x = -150, x = 150 \)

The only critical number in the domain is \( x = 150 \)
We suspect that $s(x)$ will have absolute min at $x = 150$

Double check

Consider concavity.
Study $S''(x)$

\[
S'(x) = 10 - \frac{25 \cdot 9000}{x^2}
\]

\[
S''(x) = \cdots = 0 + \frac{25 \cdot 9000 \cdot 2}{x^3}
\]

Notice that for all $x > 0$, $x^3$ will be positive.

So, for all $x > 0$, $S''(x) = \frac{25 \cdot 9000 \cdot 2}{x^3} > 0$

So graph of $S$ is concave up for all $x > 0$
So there must be an absolute min at \( x = 150 \).

So \( x = 150 \)

Now we need to find \( y \).

We know \( xy = 9000 \)

So \( y = \frac{9000}{x} = \frac{900}{15} = 60 \)

\( y = 60 \)

[End of lecture]