Section 4.2 Concavity

Definition of Concavity at a particular x-value

Words: f is concave up at $x = c$

Meaning:
- f has a tangent line at $x = c$
- for x-values near c, the graph of f stays above the tangent line.

Graphical Examples:

- Concave up at $x = c$
- Concave up at $x = c$
- No concavity at $x = c$ because no tangent line
Definition of concavity on an interval

Words: \( f \) is concave up on the interval \((a, b)\)

Meaning: for every \( x = c \) where \( a < c < b \),
\( f \) is concave up at \( c \).

\[ y = 13 \]
\[ y = 7 \]

inflection point at \((x, y) = (5, 7)\)

and \((x, y) = (10, 13)\)

But no inflection point at \( x = 15 \) because there is no point there.
Relationship between 1st derivative & concavity

Sketch graph of $f'$

- $f' \uparrow \quad f' \uparrow$
- $y$ small pos \quad $y$ large pos
- $y$ med pos
- $m$ small pos \quad $m$ media pos \quad $m$ large pos
- $m$ large neg
- $m$ med neg
- $m$ small neg

$y$ neg close to zero
$y$ med neg
$y$ large neg

$f' \uparrow \quad f' \uparrow$
Relationship between 1st derivative & concavity:

- $f'$ increasing on $(a,b) \implies f$ concave up on $(a,b)$
- $f'$ decreasing on $(a,b) \implies f$ concave down on $(a,b)$
- $f'$ constant on $(a,b) \implies f$ straight line on $(a,b)$

Class Drill 18
Class Drill 18: Using a graph of $f'$ to get information about $f$

The graph of $f'$ is shown at right.

(Note: This is not the graph of $f$.)

In this class drill, you will analyze the behavior of $f'$ and use that information to write conclusions about the behavior of $f$, which is not shown.

Using those conclusions about $f$, you will then sketch a graph of $f$.

(A) Fill in the table below.

<table>
<thead>
<tr>
<th>$x$</th>
<th>sign of $f'$ (circle one)</th>
<th>incr/decr behavior of $f'$ (circle one)</th>
<th>conclusions about behavior of function $f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x &lt; -3$</td>
<td>pos neg zero</td>
<td>incr decr horiz tan</td>
<td>$f$ increasing, concave down</td>
</tr>
<tr>
<td>$x = -3$</td>
<td>pos neg zero</td>
<td>incr decr horiz tan</td>
<td>$f$ horiz tangent, inflection</td>
</tr>
<tr>
<td>$-3 &lt; x &lt; 1$</td>
<td>pos neg zero</td>
<td>incr decr horiz tan</td>
<td>$f$ increasing, concave up</td>
</tr>
<tr>
<td>$x = 1$</td>
<td>pos neg zero</td>
<td>incr decr horiz tan</td>
<td>$f$ increasing, inflection</td>
</tr>
<tr>
<td>$1 &lt; x &lt; 4$</td>
<td>pos neg zero</td>
<td>incr decr horiz tan</td>
<td>$f$ increasing, concave down</td>
</tr>
<tr>
<td>$x = 4$</td>
<td>pos neg zero</td>
<td>incr decr horiz tan</td>
<td>$f$ horiz tangent, concave down</td>
</tr>
<tr>
<td>$4 &lt; x$</td>
<td>pos neg zero</td>
<td>incr decr horiz tan</td>
<td>$f$ decreasing, concave down</td>
</tr>
</tbody>
</table>

(B) Sketch a possible graph of $f$ at right.
Notice that it is useful to know when \( f' \) is increasing or decreasing because that tells us about concavity behavior of \( f \).

So if we study the derivative of \( f' \) and see when it is positive or negative, that will tell us when \( f' \) is increasing/decreasing that will tell us when \( f \) is concave up/down.

So we want to study the derivative of \( f' \).
This brings us to **The Second derivative**.

**Definition**

Symbol: \( f''(x) \)

Spoken: the second derivative of \( f \).

Meaning: \( \frac{d}{dx} \left( \frac{df}{dx} \right) \) that is \( \frac{d^2 f}{dx^2} \)

Alternate symbol: \( \frac{d^2 f(x)}{dx^2} \)
Examples of second derivatives

Example #1  \( f(x) = \ln(x^2 + 6x + 13) \)  find \( f''(x) \)

Solution:
\[
f'(x) = \frac{d}{dx} \ln(x^2 + 6x + 13)
\]
\[
= \frac{1}{x^2 + 6x + 13} \cdot (2x + 6)
\]
\[
= \frac{2x + 6}{x^2 + 6x + 13}
\]

Chain rule details:
inner(x) = \( x^2 + 6x + 13 \)
inner'(x) = \( 2x + 6 \)
outer( ) = \( \ln( ) \)
outer'( ) = \( \frac{1}{( )} \)
\[ f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx} \frac{2x+6}{x^2+6x+13} \]

\[
= \frac{\left(\frac{d}{dx}(2x+6)\right)(x^2+6x+13) - (2x+6)\left(\frac{d}{dx}(x^2+6x+13)\right)}{(x^2+6x+13)^2}
\]

\[
= \frac{2(x^2+6x+13) - (2x+6)(2x+6)}{(x^2+6x+13)^2}
\]

\[
= \frac{2x^2+12x+26 - (4x^2+24x+36)}{(x^2+6x+13)^2}
\]

\[
= \frac{-2x^2-12x-10}{(x^2+6x+13)^2} = \frac{-2(x^2+6x+5)}{(x^2+6x+13)^2} = f''(x)
\]

End of Lecture