Finding derivatives analytically

Remember

\[ f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \]

= derivative of \( f \)

at \( x = 2 \)

= slope of the line

tangent to graph

of \( f \) at the

point \( (x, y) = (2, f(2)) \)

More generally

\[ f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \]

= derivative of \( f \)

at \( x = a \)

= slope of the line

tangent to graph of \( f \)

at the point \( (x, y) = (a, f(a)) \)
We could use "x" instead of "a".

\[
\frac{f'(x)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \text{derivative of } f \text{ at } x
\]

The Definition of the Derivative = function that, given an actual value for x, will give the number that is the slope of the line tangent to graph of f at that x-value.

Today: Some examples of finding f'(x) using the Definition of the Derivative.
Example 1: \( f(x) = x^2 - 2x - 3 \)

Find \( f'(x) \) using the definition of the derivative.

Solution:

We need to build this: \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \)

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

\[
= \lim_{h \to 0} \frac{(x^2 + 2xh + h^2 - 2x - 2h - 3) - (x^2 - 2x - 3)}{h}
\]

**Details**

\( f(x) = x^2 - 2x - 3 \)

\( f(x) = (\_\_\_) - 2(\_\_) - 3 \) empty version

\( f(x+h) = (x+h)^2 - 2(x+h) - 3 \)

\( f(x+h) = x^2 + 2xh + h^2 - 2x - 2h - 3 \)
\[ \lim_{{h \to 0}} \frac{{2xh + h^2 - 2h}}{h} \]

\[ = \lim_{{h \to 0}} \frac{h(2x + h - 2)}{h} \]

\[ = \lim_{{h \to 0}} (2x + h - 2) \]

\[ = 2x + 0 - 2 \]

\[ = 2x - 2 \]

Conclusion: \( f'(x) = 2x - 2 \)
Remark: \( f(x) = x^2 - 2x - 3 = (x+1)(x-3) \)

- Parabola facing up
- \( y \)-intercept: \((0, -3)\)
- \( x \)-intercepts: \((1, 0)\) and \((3, 0)\)

\[ f'(x) = 2x - 2 \]

- Line
  - Slope \( m = 2 \)
  - \( y \)-intercept: \((0, -2)\)

This is the analytic version of the graphical problem that we did yesterday!

(In Class Drill 6)
Example #2 \( f(x) = \sqrt{x} \). Find \( f'(x) \) using the definition of the derivative.

\[
\frac{f'(x)}{\frac{\sqrt{x+h} - \sqrt{x}}{h}} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

Definition of the Derivative

\[
= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}
\]

Trick

\[
= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}
\]

\[
= \lim_{h \to 0} \frac{\sqrt{x+h} \sqrt{x+h} - \sqrt{x} \sqrt{x}}{h (\sqrt{x+h} + \sqrt{x})}
\]

\[
= \lim_{h \to 0} \frac{x+h - x}{h (\sqrt{x+h} + \sqrt{x})}
\]

\[
= \lim_{h \to 0} \frac{x + h - x}{h (\sqrt{x+h} + \sqrt{x})}
\]
\[
\lim_{{h \to 0}} \frac{h}{{h(\sqrt{x+h} + \sqrt{x})}}
\]

Since \(h \to 0\) we know \(h \neq 0\), so we can cancel \(\frac{h}{h}\).

\[
\lim_{{h \to 0}} \frac{1}{\sqrt{x+h} + \sqrt{x}}
\]

We can substitute \(h = 0\) because nothing goes wrong.

\[
= \frac{1}{\sqrt{x} + \sqrt{x}}
\]

\[
= \frac{1}{2\sqrt{x}}
\]

Conclusion: for \(f(x) = \sqrt{x}\) the derivative is \(f'(x) = \frac{1}{2\sqrt{x}}\).