Vector spaces $V \xrightarrow{h} W \xrightarrow{g} X$

$h: V \rightarrow W$ input $\vec{v} \in V$, output $h(\vec{v}) \in W$

$g: W \rightarrow X$ input $\vec{w} \in W$, output $g(\vec{w}) \in X$

The composition $g \circ h$ is "$g$ composed with $h$" is a map

$g \circ h: V \rightarrow X$

defined by $(g \circ h)(\vec{v}) = g(h(\vec{v}))$

Lemma 2.1 (page 226)
If $h: V \rightarrow W$ is linear and $g: W \rightarrow X$ is linear then $g \circ h$ is linear

Proof: Group work
Since $goh : V \rightarrow X$ is linear, it is natural to ask what a matrix representation of the map would be.

Suppose

<table>
<thead>
<tr>
<th>Input space</th>
<th>Output space</th>
</tr>
</thead>
<tbody>
<tr>
<td>vector space $V$</td>
<td>$X$</td>
</tr>
<tr>
<td>Basis $B$</td>
<td>$\mathcal{D}$</td>
</tr>
<tr>
<td>dimension $n$</td>
<td>$M$</td>
</tr>
</tbody>
</table>

What will be the matrix $\text{Rep}_{B,\mathcal{D}}(goh)$?

First, address the shape of the matrix.

Number of rows = dimension of output space = $m$

Number of columns = dimension of input space = $n$

This makes sense, because given an input vector $\vec{v} \in V$, the representation of $\vec{v}$ in basis $B$ would be $\text{Rep}_B(\vec{v}) = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$.
The representation of the map \( g_0h \) would be

\[
\text{Rep}_{B,D}(g_0h) = M
\]

To use these matrices to get a representation of the output \((g_0h)(\vec{v})\), we would use the representation equation

\[
\text{Rep}_D((g_0h)(\vec{v})) = \text{Rep}_{B,D}(g_0h) \cdot \text{Rep}_B(\vec{v})
\]

Matrix \( M \times n \) \( \times \) \( n \times n \) column vector \( n \times 1 \)

Product will be \( M \times 1 \) column vector.
Now that we have figured out the shape of the matrix $\text{Rep}_{\mathcal{B}, \mathcal{D}}(g; h)$, the big question is, what is the actual matrix?

It turns out that in addition to the given bases for $V$ and $X$, we also need a basis for $W$. So suppose that we have the following given information.

Vector Spaces $\xrightarrow{h} W \xrightarrow{s} X$

Bases $B \quad C$

Dimensions $n \quad r \quad m$

The big result is Theorem 2.6 (page 228)

$$\text{Rep}_{\mathcal{B}, \mathcal{D}}(g; h) = \text{Rep}_{\mathcal{C}, \mathcal{D}}(g) \cdot \text{Rep}_{\mathcal{B}, \mathcal{C}}(h)$$

These matrices will be $m \times n$. The product of these matrices will be $m \times n$. [Diagram of matrix multiplication]
Study the proof of this theorem in the book. It is tedious, but you will learn a lot.

Properties of matrix operations:

Matrix addition is **commutative and associative** (but the matrices have to be the same shape).

Matrix multiplication is **not commutative** (does not always exist when A, B are switched).

Is \((AB)C = A(BC)\) for matrix multiplication?
The fact is that matrix multiplication is associative.
The proof that \((AB)C = A(BC)\) for matrix multiplication is associative is an extremely messy proof if you use the definition of matrix multiplication (in terms of summations).

But we have another, much cleaner, proof available.

We know that every matrix can be thought of as a representation of a linear map.

Suppose that matrices \(A, B, C\) represent linear maps \(F, g, h\).

Then matrix \((AB)C\) represents linear map \((fog)oh\) while matrix \(A(BC)\) represents linear map \(f\circ(goh)\).

It is a fact that function composition is associative, so \((fog)oh = f\circ(goh)\).

This proves that \((AB)C = A(BC)\)
Proof that function composition is associative:

\[(f \circ (g \circ h))(x) = (f \circ g)(h(x))\] definition of composition

\[= f(g(h(x)))\] definition of composition

while

\[(f \circ (g \circ h))(x) = f((g \circ h)(x))\] definition of composition

\[= f(g(h(x)))\] definition of composition.

Since the outputs are always the same, we conclude that

\[(f \circ g)h = f \circ (g \circ h)\]

End of Lecture