More from Section Three. IV. 1

The Transpose

Definition of transpose

Symbol: $M^T$
Usage: $M$ is a matrix
Spoken: the transpose of $M$
Meaning: $M^T$ is the matrix that results from interchanging the rows and columns of $M$.
($1^{st}$ row becomes $1^{st}$ column, etc.)

Formula for $M^T$: If $M$ is an $m \times n$ matrix, then $M^T$ is the $n \times m$ matrix whose entries are $(M^T)_{ij} = M_{ji}$ for $i = 1 \ldots n$ and $j = 1 \ldots m$

Example 1: If $M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ then $M^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$
Example #2 If \( M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \) then \( M^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \)

Notice if \( M \) is \( m \times n \) matrix, then \( M^T \) will be \( n \times m \)

Function Notation for the transpose, \((\,)^T: M_{m \times n} \rightarrow M_{n \times m}\)

Obvious questions: ① Is \((\,)^T\) linear? (homomorphism?)
③ Is \((\,)^T\) one-to-one? yes
④ Is \((\,)^T\) onto?

Cool observation \(( (M)^T )^T = M\)

in function notation \( ( f \circ (\,)^T )^T = \text{identity map} \)

\(( ( f )^T )^T = \text{id} \)
So the transpose map is its own inverse. Functions can only have inverses if they are one-to-one and onto. So transpose must also be onto.

More Direct proof that transpose is onto

Given any desired output matrix $Y \in \mathbb{R}^{m \times n}$

Use input matrix $X = Y^T$

Then notice that the resulting output will be

$$(X^T)^T = (Y^T)^T = Y$$

We have found an input $X$ such that $(X)^T = Y$.

So the transpose map is onto.
Group work: Prove that \((C)^T\) is linear. (using abstract symbols)

Prove that if \(A, B \in \mathbb{M}_{m \times n}\) and \(\gamma_1, \gamma_2 \in \mathbb{R}\)

then prove that ...

\[
\left(\gamma_1 A + \gamma_2 B\right)^T = \gamma_1 A^T + \gamma_2 B^T
\]

These are matrices. Prove that they are equal by showing that their entries are all equal.

\[
\left(\left(\gamma_1 A + \gamma_2 B\right)^T\right)_{ij} = \left(\gamma_1 A + \gamma_2 B\right)_{ji} \quad \text{definition of transpose}
\]

\[
= \left(\gamma_1 A\right)_{ji} + \left(\gamma_2 B\right)_{ji} \quad \text{def of matrix addition}
\]

\[
= \gamma_1 \left(\left(A\right)_{ji}\right) + \gamma_2 \left(\left(B\right)_{ji}\right) \quad \text{def of scalar mult of matrices}
\]

\[
= \gamma_1 \left(\left(A^T\right)_{ij}\right) + \gamma_2 \left(\left(B^T\right)_{ij}\right) \quad \text{def of transpose}
\]

\[
= \left(\gamma_1 A^T\right)_{ij} + \left(\gamma_2 B^T\right)_{ij} \quad \text{def of scalar mult}
\]
So the matrices are equal:

\[ \mathbf{r}_1 \mathbf{A}^T + \mathbf{r}_2 \mathbf{B}^T = \mathbf{r}_1 \mathbf{A}^T + \mathbf{r}_2 \mathbf{B}^T \]

What is the kernel of the transpose function?  
(The null space) is the \( \mathbf{0} \) vector 
\[ \mathbf{0}_{\text{vec}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \]

What is the nullity? A basis for

More properly, the null space is \( \mathbb{E} \to \mathbb{B} \)

the trivial subspace of \( \mathbb{M}_{\text{vec}} \)

A basis for this null space is the empty set.

So the nullity of the transpose function is 0.
What is the range + rank of $(c)^T : M_{mxn} \rightarrow M_{mxm}$

What is the range, $R(c)^T$?

We know $(c)^T$ is onto so the range, the set of all outputs, is the whole codomain $M_{mxm}$.

$$R(c)^T = M_{mxm}$$

A basis for that space is

$$B = \left\langle \left( \begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{array} \right), \ldots, \left( \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 1 \end{array} \right) \right\rangle$$

This basis has $mn$ elements.

So rank of transpose map = dimension of $M_{mxm}$

$$= n \times m$$

End of lecture.