Section 1: Image

Suppose that $f$ is a function $f: A \rightarrow B$
If $x \in A$, then the symbol $f(x)$ denotes the output that results when $x$ is used as input.
Notice that $f(x) \in B$.
Another name for the output $f(x)$ is “the image of $x$ under the map $f$”.

So the input $x$ is an element of the domain $A$, while the image $f(x)$ is an element of the codomain $B$.

Examples

Example #1: For $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2$, we have the following:
- The image of 2 under the map $f$ is 4, because $f(2) = 2^2 = 4$.
- The image of $-2$ under the map $f$ is 4, because $f(-2) = (-2)^2 = 4$.

Example #2: For $g: \mathbb{R} \rightarrow \mathbb{R}$ defined by $g(x) = x^3$, we have the following:
- The image of 2 under the map $g$ is 8, because $g(2) = 2^3 = 8$.
- The image of $-2$ under the map $g$ is $-8$, because $g(-2) = (-2)^3 = -8$.

Section 2: Preimage

Suppose that $f$ is a function $f: A \rightarrow B$
If $y \in B$, then the symbol $f^{-1}(y)$ denotes the set of all inputs that will yield $y$ as an output.
Notice that $f^{-1}(y) \subset A$.
Another name for the set $f^{-1}(y) \subset A$ is “the preimage of $y$ under the map $f$”.

So the output $y$ is an element of the codomain $B$, while the preimage $f^{-1}(y)$ is a subset of the domain $A$.

Examples:

Example #1: For $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2$, we have the following:
- The preimage of 4 under the map $f$ is the set $\{-2,2\}$, because $f(2) = 4$ and $f(-2) = 4$
- The preimage of 0 under the map $f$ is the set $\{0\}$, because only $f(0) = 0$.
- The preimage of $-5$ under the map $f$ is the the empty set $\phi$, because there is no $x$ such that $x^2 = -5$
Using the notation for the preimage, we would write
- $f^{-1}(4) = \{-2,2\}$.
- $f^{-1}(0) = \{0\}$.
- $f^{-1}(-5) = \phi$.
Again notice that in each case, the preimage is a subset of the domain.

Example #2: For $g: \mathbb{R} \rightarrow \mathbb{R}$ defined by $g(x) = x^3$, we have the following:
- The preimage of 8 under the map $g$ is the set $\{2\}$, because only $g(2) = 8$.
- The preimage of 0 under the map $g$ is the set $\{0\}$, because only $g(0) = 0$. 
• The preimage of \(-5\) under map \(g\) is the set \(\{-(5)^{1/3}\}\), because \(g(-(5)^{1/3}) = \left(-(5)^{1/3}\right)^3 = -5\)

Using the notation for the preimage, we would write

- \(g^{-1}(8) = \{2\}\).
- \(g^{-1}(0) = \{0\}\).
- \(g^{-1}(-5) = \{-(5)^{1/3}\}\).

Again notice that in each case, the preimage is a subset of the domain.

**Section 3: Inverse Functions**

First, a definition

**Definition** of inverse functions:

**words:** \(f\) and \(g\) are inverse functions.

**meaning:** \(f\) and \(g\) are functions satisfy the following four conditions

- \(f: A \to B\) for some domain \(A\) and some codomain \(B\).
- \(g: B \to A\). That is, the domain and codomain of \(g\) are the reverse of what they are for \(f\).
- For all \(x \in A\), the equation \(g(f(x)) = x\) is true.
- For all \(y \in B\), the equation \(f(g(y)) = y\) is true.

The last two conditions are called the inverse relations.

**Additional terminology:** If \(f\) and \(g\) are inverse functions, we also say that \(g\) is an inverse function for \(f\), and we also say that \(f\) is an inverse function for \(g\).

**Examples**

**Example #1:** Let \(f: \mathbb{R} \to \mathbb{R}\) be \(f(x) = x^3\), and let \(g: \mathbb{R} \to \mathbb{R}\) be \(g(x) = x^{1/3}\).

Observe that \(g(f(x)) = (x^3)^{1/3} = x\).

Also observe that \(f(g(y)) = (y^{1/3})^3 = y\).

So the inverse relations are both true. Conclude that \(f\) and \(g\) are inverse functions.

**Example #2:** Let \(f: \mathbb{R} \to \mathbb{R}\) be defined by \(f(x) = x^2\). Then \(f\) does not have an inverse function. To see why, consider what happens when we try to come up with one. Let \(g: \mathbb{R} \to \mathbb{R}\) be defined by \(g(x) = \sqrt{x}\).

- Observe that \(g(f(2)) = \sqrt{2^2} = 2\), which is fine, but \(g(f(-2)) = \sqrt{(-2)^2} = 2\). So the equation \(g(f(x)) = x\) is not always true.
- Also observe that \(f(g(5)) = (\sqrt{5})^2 = 5\) which is fine, but \(f(g(-5)) = (\sqrt{-5})^2\) which does not even exist. So the equation \(f(g(y)) = y\) is not always true.

**Facts about inverse functions:**

- A function \(f: A \to B\) has an inverse function \(g: B \to A\) if and only if \(f\) is one-to-one and onto.
- The inverse function \(g\) will also be one-to-one and onto.
- If some functions \(f: A \to B\) and \(g: B \to A\) satisfy both inverse relations:
  - For all \(x \in A\), the equation \(g(f(x)) = x\) is true.
For all \( y \in B \), the equation \( f(g(y)) = y \) is true. Then it can be proven that \( f \) and \( g \) are both one-to-one and onto, so they qualify to be called inverse functions.

- Inverse functions are unique: A function can have only one inverse function.

**Additional notation:** If function \( f: A \rightarrow B \) has an inverse function, we use the symbol \( f^{-1} \) to denote the unique inverse function.

**Example:**
Let \( f: \mathbb{R} \rightarrow \mathbb{R} \) be defined by \( f(x) = x^3 \). Then \( f \) has an inverse function \( f^{-1}: \mathbb{R} \rightarrow \mathbb{R} \) defined by \( f^{-1}(y) = y^{1/3} \).

**Section 4: Using inverse notation:**
Observe that inverse notation and preimage notation look the same. This is confusing. I will discuss some examples:

**Example #1**
Let \( f: \mathbb{R} \rightarrow \mathbb{R} \) be defined by \( f(x) = x^3 \). What does the symbol \( f^{-1}(8) \) mean? There are two possibilities.

- Remember that in Section 2 above, the symbol \( f^{-1}(8) \) meant the preimage of 8 under the map \( f \). The preimage is always a set, a subset of the domain. We wrote \( f^{-1}(8) = \{2\} \). Notice that this is a set.
- But in Section 3, the symbol \( f^{-1} \) was used to denote the inverse function \( f^{-1}: \mathbb{R} \rightarrow \mathbb{R} \) defined by the formula \( f^{-1}(y) = y^{1/3} \). In that usage, the symbol \( f^{-1}(8) \) would denote the output that results when we feed the number \( y \) into the function \( f^{-1} \). That is \( f^{-1}(8) = (8)^{1/3} = 2 \). Notice that this is a number, not a set.

Which interpretation of the meaning of the symbol \( f^{-1}(8) \) is correct? Generally, if a function \( f \) has an inverse function, then we interpret the symbol \( f^{-1}(8) \) to mean the number that results when an input of 8 is fed into the inverse function. That is, we interpret the symbol \( f^{-1}(8) \) to mean a single element of the domain.

**Example #2**
Let \( f: \mathbb{R} \rightarrow \mathbb{R} \) be defined by \( f(x) = x^2 \). What does the symbol \( f^{-1}(4) \) mean? There is only one possibility.

- As in Section 2 above, the symbol \( f^{-1}(4) \) means the preimage of 4 under the map \( f \). The preimage is always a set, a subset of the domain. In this case, it is \( f^{-1}(4) = \{2, -2\} \).
- Because \( f(x) = x^2 \) has no inverse function, the symbol \( f^{-1}(4) \) cannot be interpreted in terms of an inverse function. It can only be interpreted as a symbol for a preimage.
More Examples:

[1] Define map \( f: \mathcal{P}_2 \rightarrow \mathbb{R}^3 \) by \( f(a + bx + cx^2) = \begin{pmatrix} b \\ a - b \\ b + c \end{pmatrix} \).

The book would write \( a + bx + cx^2 \mapsto \begin{pmatrix} b \\ a - b \\ b + c \end{pmatrix} \).

Find the image of each of these elements of the domain: (a) \( \vec{v}_1 = 2 - 3x + 4x^2 \) (b) \( \vec{v}_2 = x + x^2 \)

Solution: (a) The image of \( \vec{v}_1 \) is the vector \( f(\vec{v}_1) = f(2 - 3x + 4x^2) = \begin{pmatrix} 2 - (-3) \\ (-3) + 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} \).

(b) The image of \( \vec{v}_1 \) is the vector \( f(\vec{v}_2) = f(x + x^2) = \begin{pmatrix} 1 \\ 0 - 1 \\ 1 + 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \).

[2] Consider the isomorphism \( Rep_\beta: \mathcal{P}_2 \rightarrow \mathbb{R}^3 \), where \( \beta \) is the basis \( \beta = \langle \vec{w}_1, \vec{w}_2, \vec{w}_3 \rangle = \langle 1, 1 + x, 1 + x + x^2 \rangle \) for \( \mathcal{P}_2 \). Find the image of each of these elements of the domain: (a) \( \vec{v}_1 = 2 - 3x + 4x^2 \) (b) \( \vec{v}_2 = x + x^2 \).

Solution:

First note that the representation map \( Rep_\beta: \mathcal{P}_2 \rightarrow \mathbb{R}^3 \) works in the following way:

\[
Rep_\beta(c_1 \cdot \vec{w}_1 + c_2 \cdot \vec{w}_2 + c_3 \cdot \vec{w}_3) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}
\]

In order to determine the output, the input vector must be expressed as a linear combination of the basis vectors \( \vec{w}_1, \vec{w}_2, \vec{w}_3 \) from basis \( \beta \). So we start by expressing \( \vec{v}_1 \) and \( \vec{v}_2 \) as linear combinations of \( \vec{w}_1, \vec{w}_2, \vec{w}_3 \).

(a) \( \vec{v}_1 = 2 - 3x + 4x^2 = (5)(1) + (-7)(1 + x) + 4(1 + x + x^2) = 5\vec{w}_1 - 7\vec{w}_2 + 4\vec{w}_3 \)

(b) \( \vec{v}_2 = x + x^2 = (-1)(1) + (0)(1 + x) + 1(1 + x + x^2) = (-1)\vec{w}_1 + (0)\vec{w}_2 + (1)\vec{w}_3 \)

Now that we know those linear combinations, we can compute the representations.

(a) \( Rep_\beta(\vec{v}_1) = Rep_\beta(5\vec{w}_1 - 7\vec{w}_2 + 4\vec{w}_3) = \begin{pmatrix} 5 \\ -7 \\ 4 \end{pmatrix} \)

(b) \( Rep_\beta(\vec{v}_2) = Rep_\beta((-1)\vec{w}_1 + (0)\vec{w}_2 + (1)\vec{w}_3) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \)

[3] Every isomorphism has an inverse. For the isomorphism in problem [2], find \( Rep_\beta^{-1} \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} \).

Solution:

First note that the inverse map \( Rep_\beta^{-1}: \mathbb{R}^3 \rightarrow \mathcal{P}_2 \) works as the reverse of the map \( Rep_\beta \) as follows:

\[
Rep_\beta^{-1} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = c_1 \cdot \vec{w}_1 + c_2 \cdot \vec{w}_2 + c_3 \cdot \vec{w}_3
\]

Therefore,

\[
Rep_\beta^{-1} \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = 2\vec{w}_1 + (-1)\vec{w}_2 + 5\vec{w}_3 = 2(1) - (1 + x) + 5(1 + x + x^2) = 6 + 4x + 5x^2.
\]