[1] Define map \( f : \mathcal{P}_2 \rightarrow \mathcal{P}_3 \) by \( f(p(x)) = x \cdot p(x) \). For example, \( f(5 + 4x - x^2) = x \cdot (5 + 4x - x^2) \)
Which of these vectors are in the range space of \( f \)? Explain. (If you think a vector is in the range space, you have to give an example of an input vector that will produce that vector as an output.)
(a) \( \vec{v}_a = x^2 \)  
(b) \( \vec{v}_b = 2x + 13x^2 \)  
(c) \( \vec{v}_c = 2 + 13x^2 \)  
(d) \( \vec{v}_d = 0 \)  
(e) \( \vec{v}_e = 2 \)

**Solution:**
(a) Observe that \( f(x) = x \cdot x = x^2 = \vec{v}_a \), so \( \vec{v}_a = x^2 \) is in the range space.
(b) Observe that \( f(2 + 13x) = x \cdot (2 + 13x) = 2x + 13x^2 = \vec{v}_b \), so \( \vec{v}_b = 2x + 13x^2 \) is in the range.
(c) Observe that there is no polynomial \( p(x) \) such that \( f(p(x)) = x \cdot p(x) = 2 + 13x^2 = \vec{v}_c \).
So \( \vec{v}_c = 2 + 13x^2 \) is not in the range.
(d) Observe that \( f(0) = x \cdot (0) = 0 = \vec{v}_d \), so \( \vec{v}_d = 0 \) is in the range.
(e) Observe that there is no polynomial \( p(x) \) such that \( f(p(x)) = x \cdot p(x) = 2 = \vec{v}_e \).
So \( \vec{v}_e = 2 \) is not in the range.

For each map in problems [2], [3], [4], answer the following:
(a) Is the map onto? (Explain)
(b) Find the range space. (Explain) (The definition of range space is on page 192.)
(c) Find a basis for the range space. (Explain)
(d) Find the rank of the map. (Explain) (The definition of range space is on page 192.)

[2] The map \( f : \mathcal{P}_2 \rightarrow \mathcal{P}_3 \) by \( f(p(x)) = x \cdot p(x) \) that was introduced in problem [1].
(a) The map is not onto. We found, for example, that \( \vec{v}_c = 2 + 13x^2 \) is not in the range space.
(b) The range space is \( \mathcal{R}(f) = \{ax + bx^2 + cx^3 | a, b, c \in \mathbb{R} \} \)
(c) A basis for the range space could be \( \beta = \{x, x^2, x^3 \} \).
(d) Since a basis for \( \mathcal{R}(f) \) has three basis vectors, we conclude that \( \text{rank}(f) = \text{dim}(\mathcal{R}(f)) = 3 \).

[3] The differentiation map \( D : \mathcal{P}_2 \rightarrow \mathcal{P}_2 \) defined by \( D(f) = \frac{d}{dx} f(x) \)
(a) The map is not onto. Consider the desired output vector \( \vec{y} = x^2 \). There is no 2nd degree polynomial \( f \) such that \( D(f) = \vec{y} = x^2 \), because if \( f \) has degree 2 (or less), then \( D(f) \) will have degree 1 (or less).
(b) The range space is \( \mathcal{R}(D) = \{a + bx | a, b \in \mathbb{R} \} = \mathcal{P}_1 \).
(c) A basis for the range space could be \( \beta = \{1, x \} \).
(d) Since a basis for \( \mathcal{R}(D) \) has three basis vectors, we conclude that \( \text{rank}(D) = \text{dim}(\mathcal{R}(D)) = 2 \).

[4] The map \( f : \mathbb{R}^2 \rightarrow \mathcal{P}_2 \) defined by \( f \left( \begin{array}{c} a \\ b \end{array} \right) = 2bx - 5bx^2 \)
(a) The map is not onto. Consider the desired output vector \( \vec{y} = 1 \). There is no input vector \( \vec{x} = \left( \begin{array}{c} a \\ b \end{array} \right) \) such that \( f \left( \begin{array}{c} a \\ b \end{array} \right) = 2bx - 5bx^2 = 1 = \vec{y} \).
(b) The range space is \( \mathcal{R}(f) = \{b(2x - 5x^2) | b \in \mathbb{R} \} \).
(c) A basis for the range space could be \( \beta = \{2x - 5x^2 \} \).
(d) Since a basis for \( \mathcal{R}(f) \) has one basis vector, we conclude that \( \text{rank}(f) = \text{dim}(\mathcal{R}(f)) = 1 \).

[5] The map \( g : \mathcal{M}_{2 \times 2} \rightarrow \mathbb{R} \) defined by \( f \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = a + b + c + d \).
(a) The map is onto. Consider the desired output vector \( \vec{y} = r \in \mathbb{R} \). Let the input vector be \( \vec{x} = \left( \begin{array}{c} r \\ 0 \end{array} \right) \).
Observe that \( g(\vec{x}) = g \left( \begin{array}{c} r \\ 0 \end{array} \right) = r + 0 + 0 + 0 = r = \vec{y} \).
(b) The range space is \( \mathcal{R}(g) = \mathbb{R} \).
(c) A basis for the range space could be \( \beta = \{1 \} \).
(d) Since a basis for \( \mathcal{R}(g) \) has one basis vector, we conclude that \( \text{rank}(g) = \text{dim}(\mathcal{R}(g)) = 1 \).