[1] Consider the isomorphism $\text{Rep}_\beta: \mathcal{P}_2 \rightarrow \mathbb{R}^3$, where $\beta$ is the basis $\beta = \langle \overline{w}_1, \overline{w}_2, \overline{w}_3 \rangle = (1, 1 + x, 1 + x + x^2)$ for $\mathcal{P}_2$. Every isomorphism has an inverse. Find $\text{Rep}_\beta^{-1}\begin{pmatrix} 7 \\ -5 \\ 3 \end{pmatrix}$.

**Solution:** First note that the inverse map $\text{Rep}_\beta^{-1}: \mathbb{R}^3 \rightarrow \mathcal{P}_2$ works as the reverse of the map $\text{Rep}_\beta$ as follows:

$\text{Rep}_\beta^{-1}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = c_1 \cdot \overline{w}_1 + c_2 \cdot \overline{w}_2 + c_3 \cdot \overline{w}_3$. So $\text{Rep}_\beta^{-1}\begin{pmatrix} 7 \\ -5 \\ 3 \end{pmatrix} = 7\overline{w}_1 + (-1)\overline{w}_2 + 3\overline{w}_3 = 7(1) - 5(1 + x) + 3(1 + x + x^2) = 5 - 2x + 3x^2$.

[2] Decide whether each map $f$ is linear. (That is, decide if it is a homomorphism.) If it is linear, then prove it. If it is not linear, then state a condition that it fails to satisfy. Bonus: Determine if each is an isomorphism.

(a) $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $f\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$. (This is an example of a “projection map”.)

**Solution:**

Let $\overrightarrow{v}_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ and $\overrightarrow{v}_2 = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ and $r_1, r_2 \in \mathbb{R}$. We must show that $f(r_1 \overrightarrow{v}_1 + r_2 \overrightarrow{v}_2) = r_1 f(\overrightarrow{v}_1) + r_1 f(\overrightarrow{v}_2)$.

$f(r_1 \overrightarrow{v}_1 + r_2 \overrightarrow{v}_2) = f\left(r_1 \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + r_2 \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}\right)$ definition of $\overrightarrow{v}_1, \overrightarrow{v}_2$.

$= f\left(r_1 \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + r_2 \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}\right)$ definition of scalar multiplication

$= f\left(r_1 \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + r_2 \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}\right)$ definition of vector addition

$= \begin{pmatrix} r_1 a_1 + r_2 a_2 \\ r_1 b_1 + r_2 b_2 \end{pmatrix}$ definition of how $f$ works.

$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ definition of vector addition

$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ definition of scalar multiplication

$= r_1 f\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + r_2 f\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ definition of how $f$ works.

$= r_1 f(\overrightarrow{v}_1) + r_2 f(\overrightarrow{v}_2)$ definition of $\overrightarrow{v}_1, \overrightarrow{v}_2$.

**Conclude that $f$ is linear.**

**Notice that $f$ is not an isomorphism because:**

It is not one-to-one: $f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = f\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

It is not onto: Given the desired output $\overrightarrow{y} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, there is no input $\overrightarrow{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ such that $f(\overrightarrow{v}) = \overrightarrow{y}$.

(b) $f: \mathbb{R}^2 \rightarrow \mathcal{P}_2$ defined by $f\begin{pmatrix} a \\ b \end{pmatrix} = (a + b) + bx^2$.

**Solution:**

Let $\overrightarrow{v}_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ and $\overrightarrow{v}_2 = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ and $r_1, r_2 \in \mathbb{R}$. We must show that $f(r_1 \overrightarrow{v}_1 + r_2 \overrightarrow{v}_2) = r_1 f(\overrightarrow{v}_1) + r_1 f(\overrightarrow{v}_2)$.

$f(r_1 \overrightarrow{v}_1 + r_2 \overrightarrow{v}_2) = f\left(r_1 \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + r_2 \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}\right)$ definition of $\overrightarrow{v}_1, \overrightarrow{v}_2$.

$= f\left(r_1 \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + r_2 \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}\right)$ definition of scalar multiplication

$= f\left(r_1 \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + r_2 \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}\right)$ definition of vector addition

$= \begin{pmatrix} r_1 a_1 + r_2 a_2 \\ r_1 b_1 + r_2 b_2 \end{pmatrix}$ definition of how $f$ works.

$= (r_1 a_1 + r_2 a_2 + r_1 b_1 + r_2 b_2) + (r_1 b_1 + r_2 b_2)x^2$ definition of how $f$ works.

$= r_1((a_1 + b_1) + b_1 x^2) + r_2((a_2 + b_2) + b_2 x^2)$ arithmetic
\[ r_1 f \left( \frac{a_1}{b_1} \right) + r_2 f \left( \frac{a_2}{b_2} \right) \quad \text{definition of how } f \text{ works.} \\
= r_1 f \left( \overrightarrow{v_1} \right) + r_1 f \left( \overrightarrow{v_2} \right) \quad \text{definition of } \overrightarrow{v_1}, \overrightarrow{v_2}. \\
\]

Conclude that \( f \) is linear.

Notice that \( f \) is one-to-one: Suppose that \( f \left( \overrightarrow{v_1} \right) = f \left( \overrightarrow{v_2} \right) \) for some input vectors \( \overrightarrow{v_1} = \left( \frac{a_1}{b_1} \right) \) and \( \overrightarrow{v_2} = \left( \frac{a_2}{b_2} \right). \) Then \( f \left( \frac{a_1}{b_1} \right) = f \left( \frac{a_2}{b_2} \right) \), so \( a_1 + b_1 + b_1 x^2 = (a_2 + b_2) + b_2 x^2. \) Equating coefficients of \( x^2 \) tells us that \( b_1 = b_2. \) Then equating constant terms tells us that \( (a_1 + b_1) = (a_2 + b_2). \) Substituting in \( b_1 = b_2, \) tells us that \( a_1 = a_2. \) Therefore, \( \overrightarrow{v_1} = \overrightarrow{v_2}. \)

But \( f \) is not onto: Given the desired output \( \ddot{y} = x, \) there is no input \( \ddot{v} = \left( \frac{a}{b} \right) \) such that \( f(\ddot{v}) = \ddot{y}. \)

Conclude that \( f \) is not an isomorphism.

(c) \( f: \mathbb{R}^2 \rightarrow \mathbb{P}_1 \) defined by \( f \left( \frac{a}{b} \right) = ax + b. \)

Solution:

Let \( \overrightarrow{v_1} = \left( \frac{a_1}{b_1} \right) \) and \( \overrightarrow{v_2} = \left( \frac{a_2}{b_2} \right) \) and \( r_1, r_2 \in \mathbb{R}. \) We must show that \( f(r_1 \overrightarrow{v_1} + r_2 \overrightarrow{v_2}) = r_1 f(\overrightarrow{v_1}) + r_1 f(\overrightarrow{v_2}). \)

\[
f(r_1 \overrightarrow{v_1} + r_2 \overrightarrow{v_2}) = f \left( r_1 \left( \frac{a_1}{b_1} \right) + r_2 \left( \frac{a_2}{b_2} \right) \right) \quad \text{definition of } \overrightarrow{v_1}, \overrightarrow{v_2}. \\
= f \left( \frac{r_1 a_1 + r_2 a_2}{b_1 + r_2 b_2} \right) \quad \text{definition of scalar multiplication & vector addition} \\
= (r_1 a_1 + r_2 a_2) x + (r_1 b_1 + r_2 b_2) \quad \text{definition of how } f \text{ works.} \\
= r_1 (a_1 x + b_1) + r_2 (a_2 x + b_2) \quad \text{arithmetic} \\
= r_1 f \left( \frac{a_1}{b_1} \right) + r_2 f \left( \frac{a_2}{b_2} \right) \quad \text{definition of how } f \text{ works.} \\
= r_1 f(\overrightarrow{v_1}) + r_2 f(\overrightarrow{v_2}) \quad \text{definition of } \overrightarrow{v_1}, \overrightarrow{v_2}. \\
\]

Conclude that \( f \) is linear.

Notice that \( f \) is one-to-one: Suppose that \( f \left( \overrightarrow{v_1} \right) = f \left( \overrightarrow{v_2} \right) \) for some input vectors \( \overrightarrow{v_1} = \left( \frac{a_1}{b_1} \right) \) and \( \overrightarrow{v_2} = \left( \frac{a_2}{b_2} \right). \) Then \( f \left( \frac{a_1}{b_1} \right) = f \left( \frac{a_2}{b_2} \right) \), so \( a_1 x + b_1 = a_2 x + b_2. \) Equating coefficients of powers of \( x \) tells us that \( a_1 = a_2 \) and \( b_1 = b_2. \) Therefore, \( \overrightarrow{v_1} = \overrightarrow{v_2}. \) And \( f \) is onto: Given any desired output \( \ddot{y} = ax + b, \) the input \( \ddot{v} = \left( \frac{a}{b} \right) \) will give that desired output. That is, \( f(\ddot{v}) = \ddot{y}. \)

Conclude that \( f \) is isomorphism.

[3] Stating that a function is linear is different from stating that its graph is a line. The function \( f: \mathbb{R} \rightarrow \mathbb{R} \) defined by \( f(x) = 5x + 7 \) has a graph that is a line. Show that it is not a linear function

Solution:

Show that \( f \) does not preserve vector addition. Let \( \overrightarrow{v_1} = 1 \in \mathbb{R} \) and \( \overrightarrow{v_2} = 1 \in \mathbb{R}. \)
Then observe that \( f(\overrightarrow{v_1} + \overrightarrow{v_2}) = f(1 + 1) = f(2) = 5(2) + 7 = 17 \)
Then observe that \( f(\overrightarrow{v_1}) + f(\overrightarrow{v_2}) = f(1) + f(1) = (5(1) + 72) + (5(1) + 7) = 12 + 12 = 24. \)
So \( f(\overrightarrow{v_1} + \overrightarrow{v_2}) \neq f(\overrightarrow{v_1}) + f(\overrightarrow{v_2}). \)

Show that \( f \) also does not preserve scalar multiplication. Let \( \ddot{v} = 1 \) and \( c = 2. \)
Then observe that \( f(c \ddot{v}) = f(2(1)) = f(2) = 5(2) + 7 = 17 \)
Then observe that \( cf(\ddot{v}) = 2f(1) = 2 \cdot (5(1) + 7) = 2 \cdot (12) = 24. \)
So \( f(c \ddot{v}) \neq cf(\ddot{v}). \)

[4] (a) The map the “evaluation at 5 map”. That is the map \( eval_5: \mathcal{F} \rightarrow \mathbb{R} \) defined by \( eval_5(f) = f(5). \)

Solution:

Let \( \overrightarrow{v_1} = f \) and \( \overrightarrow{v_2} = g \) and \( a, b \in \mathbb{R}. \) We must show that \( eval_5(af + bg) = a \cdot eval_5(f) + b \cdot eval_5(g). \)
\[
eval_5(af + bg) = (af + bg)(5) \quad \text{definition of how } \eval_5 \text{ works}
\]
\[
= (af)(5) + (bg)(5) \quad \text{definition of function addition}
\]
\[
= a \cdot f(5) + b \cdot g(5) \quad \text{definition of scalar multiplication of functions}
\]
\[
= a \cdot \eval_5(f) + b \cdot \eval_5(g) \quad \text{definition of how } \eval_5 \text{ works}
\]

Conclude that \(f\) is linear.

Notice that \(\eval_5\) is not one-to-one:

Let \(f(x) = x^2\) and let \(g(x) = x + 20\). Then \(f \neq g\), but \(\eval_5(f) = f(5) = (5)^2 = 25\) and \(\eval_5(g) = g(5) = 20 + (5) = 25\). So \(\eval_5(f) = \eval_5(g)\) even though \(f \neq g\).

Notice that \(\eval_5\) is onto:

Given the desired output \(\bar{y} = r \in \mathbb{R}\), let \(f\) be the constant function \(f(x) = r\). Then \(\eval_5f = f(5) = r\).

Conclude that since \(\eval_5\) is not one-to-one, it is not an isomorphism.

(b) The map “Definite Integral from 0 to 1”. That is, the map \(I : C^0 \to \mathbb{R}\) defined by \(I(f) = \int_{t=0}^{t=1} f(t)dt\).

Solution: Show that the map \(I\) preserves all linear combinations.

Suppose \(\vec{v}_1 = f \in C^0\) and \(\vec{v}_2 = g \in C^0\) and \(a, b \in \mathbb{R}\). Then

\[
\begin{align*}
I(a \cdot \vec{v}_1 + b \cdot \vec{v}_2) &= I(a \cdot f + b \cdot g) \\
&= \int_{t=0}^{t=1} (af + bg)(t)dt \\
&= \int_{t=0}^{t=1} ((af)(t) + (bg)(t))dt \\
&= \int_{t=0}^{t=1} (af(t) + bg(t))dt \\
&= a \cdot \left( \int_{t=0}^{t=1} f(t)dt \right) + b \cdot \left( \int_{t=0}^{t=1} g(t)dt \right) \\
&= a \cdot I(f) + b \cdot I(g)
\end{align*}
\]

The vectors are the functions \(f, g\).

Conclude that \(I\) is linear. That is, \(I\) is a homomorphism.

Observe that \(I\) is not one-to-one

Let \(\vec{v}_1 = f\) be the function \(f(x) = 1\) and Let \(\vec{v}_2 = g\) be the function \(g(x) = 2x\). Then \(\vec{v}_1 \neq \vec{v}_2\).

But \(I(\vec{v}_1) = I(f) = \int_{t=0}^{t=1} 1dt = 1\) while \(I(\vec{v}_2) = I(g) = \int_{t=0}^{t=1} 2tdt = 1\), so \(I(\vec{v}_1) = I(\vec{v}_2)\).

Observe that \(I\) is onto

Let \(y \in \mathbb{R}\) be any desired output.

Define the input vector \(\vec{y} = f\) to be the constant function \(f(x) = y\).

Then \(I(\vec{y}) = I(f) = \int_{t=0}^{t=1} ydt = y \cdot \int_{t=0}^{t=1} dt = y \cdot 1 = y\).

We have found an input that gives the desired output.

Since \(I\) is not one-to-one, it is not an isomorphism.

[5] (a) The map “Definite Integral from 0 to \(x\)”. That is, the map \(I : C^0 \to C^1\) defined by \(I(f) = \int_{t=0}^{t=x} f(t)dt\).

Solution:
The proof that the map \(I\) preserves all linear combinations is just like the proof in [4](b). The fact that the upper limit of integration is \(t = x\) rather than \(t = 1\) does not change any of the steps or any of the justifications.

Conclude that \(I\) is linear. That is, \(I\) is a homomorphism.
Observe that $I$ is one-to-one
Suppose that $\vec{v}_1 = f$ and $\vec{v}_2 = g$ are two functions such that $I(f) = I(g)$.
As discussed in class, $I(f) = \int_{t=0}^{t=x} f(t) dt$ will be a function that is an antiderivative of $f(x)$, and $I(g)$ will be a function that is an antiderivative of $g(x)$.
So, if we take $\frac{d}{dx}$ of both sides of the equation $I(f) = I(g)$, we will obtain the new equation
$$\frac{d}{dx} I(f) = \frac{d}{dx} I(g)$$
$$f(x) = g(x)$$

Conclude that $f = g$, so $\vec{v}_1 = \vec{v}_2$.

Observe that $I$ is not onto
As discussed in class, $I(f) = \int_{t=0}^{t=x} f(t) dt$ will be a function of the variable $x$.
If we substitute $x = 0$ into this function, we obtain
$$(I(f))(0) = \int_{t=0}^{t=0} f(t) dt = 0$$
because the lower and upper limits of integration are the same.
So $I(f) = \int_{t=0}^{t=x} f(t) dt$ will not only be a function that is an antiderivative of $f(x)$, it will be the particular antiderivative that has value $0$ at $x = 0$. (In calculus, you sometimes denoted an antiderivative of $f$ by $F$. So $I(f)$ is the particular antiderivative $F(x)$ that has the property that $F(0) = 0$.
So consider the desired output vector $\vec{y}$ that is the function $\cos(x)$. observe that this desired output function has the property that $\cos(0) = 1$, not $0$. Therere, we know that there is no input vector $\vec{v}$, that is no input function $f$, such that $I(\vec{v}) = \vec{y}$.

Since $I$ is not onto, it is not an isomorphism.

(b) The “Derivative” map That is, the map $D: C^1 \rightarrow C^0$ defined by $D(f) = \frac{df}{dx}$.

Solution:
Show that $D$ preserves linear combinations. Suppose $\vec{v}_1 = f \in C^1$ and $\vec{v}_2 = g \in C^1$ and $a, b \in \mathbb{R}$. Then
$$D(a\vec{v}_1 + b\vec{v}_2) = D(af + bg)$$
by the definition of $D$
$$= \frac{d}{dx}((af + bg)(x))$$
by the definition of function addition
$$= \frac{d}{dx}((af)(x) + (bg)(x))$$
by the definition of scalar multiplication
$$= \frac{d}{dx}(a \cdot f(x) + b \cdot g(x))$$
by the Sum Rule & Constant multiple rules for Derivatives
$$= a \frac{df}{dx} + b \frac{dg}{dx}$$
by the definition of $D$
$$= aD(f) + bD(g)$$
The vectors are the functions $f, g$.

Conclude that $D$ is linear. That is, $D$ is a homomorphism.

Notice that $D$ is not one-to-one: Let $f(x) = x^2$ and let $g(x) = x^2 + 20$. Then $f \neq g$, but $D(f) = 2x$ and $D(g) = 2x$.So $D(f) = D(g)$ even though $f \neq g$.

Notice that $D$ is onto:
Given the desired output $\vec{y} = f \in C^0$ let $F$ be an antiderivative for $f$. (We know that such an antiderivative exists because $f$ is continuous. In fact, we know that one way to get an antiderivative $F$ is by defining $F = I(f)$, where $I$ is The map “Definite Integral from $0$ to $x$”. That is, the map $I: C^0 \rightarrow C^1$ defined by $I(f) = \int_{t=0}^{t=x} f(t) dt$ from part (a) of this sproblem

Conclude that since $D$ is not one-to-one, it is not an isomorphism.