[1] Define map \( f: \mathbb{P}_2 \to \mathbb{R}^3 \) by \( f(a + bx + cx^2) = \left( \begin{array}{c} 2c \\ b-a \\ c-b \end{array} \right) \). The book would write \( a + bx + cx^2 \to \left( \begin{array}{c} 2c \\ b-a \\ c-b \end{array} \right) \).

Find the image of each of these elements of the domain: (a) \( \overrightarrow{v}_1 = 4 - 3x + 2x^2 \) (b) \( \overrightarrow{v}_2 = x + x^2 \)

Solution: (a) The image of \( \overrightarrow{v}_1 \) is the vector \( f(\overrightarrow{v}_1) = f(4 - 3x + 2x^2) = \left( \begin{array}{c} 2(2) \\ -3 - 4 \\ 2 - (-3) \end{array} \right) = \left( \begin{array}{c} 4 \\ -7 \\ 5 \end{array} \right) \).

(b) The image of \( \overrightarrow{v}_2 \) is the vector \( f(\overrightarrow{v}_2) = f(x + x^2) = \left( \begin{array}{c} 2(1) \\ 1 - 0 \\ 1 - 1 \end{array} \right) = \left( \begin{array}{c} 2 \\ 1 \\ 0 \end{array} \right) \).

[2] Consider the isomorphism \( \text{Rep}_\beta: \mathbb{P}_2 \to \mathbb{R}^3 \), where \( \beta = (w_1, w_2, w_3) = (1, 1 + x, 1 + x + x^2) \)

for \( \mathbb{P}_2 \). Find the image of each of these elements of the domain: (a) \( \overrightarrow{v}_1 = 7 - 5x + 3x^2 \) (b) \( \overrightarrow{v}_2 = x + x^2 \).

Solution:

First note that the representation map \( \text{Rep}_\beta: \mathbb{P}_2 \to \mathbb{R}^3 \) works in the following way:

\[
\text{Rep}_\beta(c_1 \cdot \overrightarrow{w}_1 + c_2 \cdot \overrightarrow{w}_2 + c_3 \cdot \overrightarrow{w}_3) = \left( \begin{array}{c} c_1 \\ c_2 \\ c_3 \end{array} \right)
\]

In order to determine the output, the input vector must be expressed as a linear combination of the basis vectors \( \overrightarrow{w}_1, \overrightarrow{w}_2, \overrightarrow{w}_3 \) from basis \( \beta \). So we start by expressing \( \overrightarrow{v}_1 \) and \( \overrightarrow{v}_2 \) as linear combinations of \( \overrightarrow{w}_1, \overrightarrow{w}_2, \overrightarrow{w}_3 \).

(a) \( \overrightarrow{v}_1 = 7 - 5x + 3x^2 = (12)(1) + (-8)(1 + x) + 3(1 + x + x^2) = 12\overrightarrow{w}_1 - 8\overrightarrow{w}_2 + 3\overrightarrow{w}_3 \)

(b) \( \overrightarrow{v}_2 = x + x^2 = (-1)(1) + (0)(1 + x) + 1(1 + x + x^2) = -\overrightarrow{w}_1 + \overrightarrow{w}_2 + \overrightarrow{w}_3 \)

Now that we know those linear combinations, we can compute the representations.

(a) \( \text{Rep}_\beta(\overrightarrow{v}_1) = \text{Rep}_\beta(5\overrightarrow{w}_1 - 7\overrightarrow{w}_2 + 4\overrightarrow{w}_3) = \left( \begin{array}{c} 12 \\ -8 \\ 3 \end{array} \right) \)

(b) \( \text{Rep}_\beta(\overrightarrow{v}_2) = \text{Rep}_\beta((-1)\overrightarrow{w}_1 + 0\overrightarrow{w}_2 + 1\overrightarrow{w}_3) = \left( \begin{array}{c} -1 \\ 0 \\ 1 \end{array} \right) \)

[3] Decide whether each map \( f \) is an isomorphism. If it is an isomorphism, then prove it. If it is not an isomorphism, then state a condition that it fails to satisfy.

(a) \( f: \mathcal{M}_{2 \times 2} \to \mathbb{R} \) defined by \( f(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = bc \).

Solution: This map is not an isomorphism. (Note that showing any one of these failures would be sufficient.)

\( f \) is not one-to-one. To see why, let \( \overrightarrow{v}_1 = \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) \) and \( \overrightarrow{v}_2 = \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) \).

Then observe that \( \overrightarrow{v}_1 \neq \overrightarrow{v}_2 \) and yet \( f(\overrightarrow{v}_1) = f(\overrightarrow{v}_2) = 0 \cdot 1 = 0 = 1 \cdot 0 = f(0 \cdot 1) = f(0 \cdot 0) = f(\overrightarrow{v}_2) \).

\( f \) also does not preserve vector addition. To see why, let \( \overrightarrow{v}_1 = \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) \) and \( \overrightarrow{v}_2 = \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) \).

Then observe that \( f(\overrightarrow{v}_1 + \overrightarrow{v}_2) = f(\left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) + \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right)) = f(\left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right)) = 1 \cdot 1 = 1 

Then observe that \( f(\overrightarrow{v}_1) + f(\overrightarrow{v}_2) = f(\left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right)) + f(\left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right)) = 1 \cdot 0 + 0 \cdot 1 = 0 + 0 = 0 \).

So \( f(\overrightarrow{v}_1 + \overrightarrow{v}_2) \neq f(\overrightarrow{v}_1) + f(\overrightarrow{v}_2) \).
**f also does not preserve scalar multiplication.** To see why, let \( \vec{v} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and \( c = 2 \).

Then observe that \( f(c\vec{v}) = f \left( 2 \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right) = f \left( \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \right) = 2 \cdot 2 = 4 \).

Then observe that \( cf(\vec{v}) = 2f \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) = 2 \cdot (1 \cdot 1) = 2. \)

So \( f(c\vec{v}) \neq cf(\vec{v}). \)

**(b) \( f: \mathcal{M}_{2 \times 2} \rightarrow \mathbb{R}^4 \)** defined by \( f \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} a \\ b - a \\ c - b \\ d - c \end{array} \right). \)

**Solution: This map is an isomorphism.**

**f is one-to-one.**

To see why, suppose \( \vec{v}_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} \) and \( \vec{v}_2 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix} \) and suppose that \( f(\vec{v}_1) = f(\vec{v}_2). \)

Then \( \begin{pmatrix} a_1 \\ b_1 - a_1 \\ c_1 - b_1 \\ d_1 - c_1 \end{pmatrix} = \begin{pmatrix} a_2 \\ b_2 - a_2 \\ c_2 - b_2 \\ d_2 - c_2 \end{pmatrix}. \)

The top entries must match, so \( a_1 = a_2. \)

Replacing \( a_1 \) with \( a_2 \) in the second entry and cancelling yields \( b_1 = b_2. \)

Similary, we can show that \( c_1 = c_2 \) and \( d_1 = d_2. \)

Conclude that \( \vec{v}_1 = \vec{v}_2. \)

**f is onto.** To see why, suppose that \( \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \) is any desired output in \( \mathbb{R}^4. \)

Then let \( \vec{x} = \begin{pmatrix} y_1 & (y_2 + y_1) \\ y_3 + y_2 + y_1 & y_4 + y_3 + y_2 + y_1 \end{pmatrix}. \)

Observe that

\[
f(\vec{x}) = f \left( \begin{array}{cc} y_1 \\ y_3 + y_2 + y_1 \end{array} \right) \left( \begin{array}{cc} (y_2 + y_1) \\ y_4 + y_3 + y_2 + y_1 \end{array} \right) = \begin{pmatrix} y_1 \\ y_2 + y_1 \\ y_3 + y_2 + y_1 \\ y_4 + y_3 + y_2 + y_1 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \vec{y}\)

We have found an \( \vec{x} \) such that \( f(\vec{x}) = \vec{y}. \)

**f preserves vector addition.** To see why, suppose \( \vec{v}_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} \) and \( \vec{v}_2 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix}. \) Then

\[
f(\vec{v}_1 + \vec{v}_2) = f \left( \begin{array}{cc} (a_1 + a_2) \\ (c_1 + c_2) \\ (d_1 + d_2) \end{array} \right) = \begin{pmatrix} b_1 + b_2 \\ (c_1 + c_2) - (b_1 + b_2) \\ (d_1 + d_2) - (c_1 + c_2) \end{pmatrix} = \begin{pmatrix} b_1 + a_1 \\ (c_1 + c_2) - (b_1 + b_2) \\ (d_1 + d_2) - (c_1 + c_2) \end{pmatrix} = \begin{pmatrix} b_1 - a_1 \\ c_1 - b_1 \\ d_1 - c_1 \end{pmatrix} + \begin{pmatrix} b_2 - a_2 \\ c_2 - b_2 \\ d_2 - c_2 \end{pmatrix} = f \left( \begin{array}{cc} a_1 \\ c_1 \\ d_1 \end{array} \right) + f \left( \begin{array}{cc} a_2 \\ c_2 \\ d_2 \end{array} \right) = f(\vec{v}_1) + f(\vec{v}_2)\)

**f preserves scalar multiplication.** To see why, suppose \( \vec{v} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \) and \( r \in \mathbb{R}. \) Then
\[
f(r \vec{v}) = f \left( r \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = f \left( r \begin{pmatrix} ra \\ rc \end{pmatrix} \begin{pmatrix} rb \\ rd \end{pmatrix} = \begin{pmatrix} ra \\ rb - ra \\ rc - rb \\ rd - rc \end{pmatrix} \right) = r \begin{pmatrix} ra \\ r(b - a) \\ r(c - b) \\ r(d - c) \end{pmatrix} = r \begin{pmatrix} a \\ b - a \\ c - b \\ d - c \end{pmatrix} = rf(\vec{v})
\]

(e) \( f: M_{2 \times 2} \rightarrow P_3 \) defined by \( f \left( \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \right) = 1 + (a + b)x + (b + c)x^2 + (c + d)x^3 \).

**Solution:** This map is not an isomorphism. To see why, observe that \( f \left( \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right) = 1 \neq \overline{0}, \) so we know that \( f \) cannot preserve vector space operations.

[4] (a) The function \( f: \mathbb{R} \rightarrow \mathbb{R} \) defined by \( f(x) = x^3 \) is not an isomorphism. Why not? (Explain which of the isomorphism requirements the function fails.)

**Solution:** (Any one of these failures would be sufficient to prove that \( f \) is not an isomorphism.)

- **\( f \) does not preserve vector addition.** To see why, let \( \vec{v}_1 = 1 \) and \( \vec{v}_2 = 1 \).
  
  Then observe that \( f(\vec{v}_1 + \vec{v}_2) = f(1 + 1) = f(2) = 8 \)
  
  and observe that \( f(\vec{v}_1) + f(\vec{v}_2) = f(1) + f(1) = 1 + 1 = 2 \).
  
  So \( f(\vec{v}_1 + \vec{v}_2) \neq f(\vec{v}_1) + f(\vec{v}_2) \).

- **\( f \) also does not preserve scalar multiplication.** To see why, let \( \vec{v} = 1 \) and \( c = 2 \).
  
  Then observe that \( f(c \vec{v}) = f(2 \cdot 1) = f(2) = 8 \)
  
  and observe that \( cf(\vec{v}) = 2f(1) = 2 \cdot 1 = 2 \).
  
  So \( f(c \vec{v}) \neq cf(\vec{v}) \).

(b) The function \( f: \mathbb{R} \rightarrow \mathbb{R} \) defined by \( f(x) = e^x \) is not an isomorphism. Why not? (Explain which of the isomorphism requirements the function fails.)

**Solution:** (Any one of these failures would be sufficient to prove that \( f \) is not an isomorphism.)

- **\( f \) is not onto.** To see why, let \( y = -1 \). There is no \( x \) such that \( f(x) = y \).

  **\( f \) does not preserve vector addition.** To see why, let \( \vec{v}_1 = 0 \) and \( \vec{v}_2 = 0 \).
  
  Then observe that \( f(\vec{v}_1 + \vec{v}_2) = f(0 + 0) = f(0) = e^0 = 1 \)
  
  and observe that \( f(\vec{v}_1) + f(\vec{v}_2) = f(0) + f(0) = e^0 + e^0 = 1 + 1 = 2 \).
  
  So \( f(\vec{v}_1 + \vec{v}_2) \neq f(\vec{v}_1) + f(\vec{v}_2) \).

  **\( f \) also does not preserve scalar multiplication.** To see why, let \( \vec{v} = 0 \) and \( c = 2 \).
  
  Then observe that \( f(c \vec{v}) = f(2 \cdot 0) = f(0) = e^0 = 1 \)
  
  and observe that \( cf(\vec{v}) = 2f(0) = 2 \cdot e^0 = 2 \cdot 1 = 2 \).
  
  So \( f(c \vec{v}) \neq cf(\vec{v}) \).

(c) Give an example of a function \( f: \mathbb{R} \rightarrow \mathbb{R} \) that is onto but not one-to-one. Your function must be unique, not the same function as anybody else in the class.

**Solution:**

We discussed one example in class: the cubic function \( f(x) = (x - a)(x - b)(x - c) \) with \( a, b, c \) not identical would be onto but not one-to-one. Observe that because it is a cubic polynomial with positive leading coefficient, its graph goes down on the left and up on the right (so it is ONTO), and the graph has \( x \) intercepts at \( x = a \) and \( x = b \) and \( x = c \) (so it is not one-to-one). Choose any three numbers \( a, b, c \) not identical to build an actual example of such a function. My example is unique in all the world:

\[ f(x) = (x - 13)(x - 17)(x - 19) \]
(d) Give an example of a function \( f: \mathbb{R} \to \mathbb{R} \) that is an isomorphism. Again, your function must be unique.

**Solution:**

As we discussed in class, for a function \( f: \mathbb{R} \to \mathbb{R} \) to preserve vector space operations (vector addition and scalar multiplication of vectors) the function will have to be of the form \( f(x) = kx \) for some constant \( k \). (You will prove this in problem [5].) The additional requirement that \( f \) be one-to-one means that the constant \( k \) must be non-zero (see my proof below). Choose any constant \( k \neq 0 \) to build an actual example of such a function \( f(x) = kx \). My example is unique in all the world:

\[
 f(x) = 54.17385x
\]

[5] Prove that if \( f: \mathbb{R} \to \mathbb{R} \) is an isomorphism then \( f \) must be of the form \( f(x) = kx \) where \( k \) is some non-zero real number. That is, prove that if \( f: \mathbb{R} \to \mathbb{R} \) is an isomorphism then there exists a non-zero real number \( k \) such that \( f(x) = kx \).

**Proof:** (The first part of this proof was done in class.)

Suppose that \( f: \mathbb{R} \to \mathbb{R} \) is an isomorphism.

Then \( f \) passes the four isomorphism tests.

**Use the fact that \( f \) passes test 2**

In particular, \( f \) passes test 2. That is, \( f \) preserves scalar multiplication.

That means that the equation \( f(rx) = rf(x) \) is true for all real numbers \( r \) and \( x \).

In particular, the equation, is true when \( x = 1 \).

So the equation \( f(r) = rf(1) \) is true for all real numbers \( r \).

We can switch the order on the right side: The equation \( f(r) = f(1)r \) is true for all real numbers \( r \).

And we can change the order: The equation \( f(t) = f(1)t \) is true for all real numbers \( t \).

We can even use the letter \( x \): The equation \( f(x) = f(1)x \) is true for all real numbers \( x \).

Let \( k \) be the real number \( k = f(1) \).

Observe that we have found a real number \( k \) such that \( f(x) = kx \) is true for all real numbers \( x \).

**Use the fact that \( f \) passes test 3 to prove that \( k = f(1) \neq 0 \).**

So far, we have found our number \( k \), but we have not proven that \( k \) is non-zero.

Now we must prove that \( k \) is non-zero. But \( k \) was defined as \( k = f(1) \), so we must prove that \( f(1) \neq 0 \).

We will prove this by contradiction. That is, we will assume that \( f(1) = 0 \), and show that that leads to a contradiction.

Assume that \( f(1) = 0 \).

Then \( f(2) = f(2 \cdot 1) = 2f(1) = 2(0) = 0 \).

So even though \( 1 \neq 2 \), it would turn out that \( f(1) = f(2) \).

This would tell us that \( f \) is not one-to-one.

But that would contradict the fact that \( f \) is known to be one-to-one, because \( f \) passes test 3.

Therefore our assumption that \( f(1) = 0 \) was incorrect.

Conclude that \( f(1) \) must not be zero.

We have proven that \( k = f(1) \neq 0 \).

**Conclusion**

We have found a non-zero real number \( k \) such that \( f(x) = kx \) is true for all real numbers \( x \).

**End of Proof**