Exercises using Section Two.I.1 Concepts:

[1] Let \( S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x - 5y + 7z = 0 \right\} \). Show that \( S \) is a vector space.

(Use only Section Two.I.1 concepts. That is, prove that each of the ten vector space conditions is satisfied.)

[2] Is the set \( \mathbb{Z} \) of integers a vector space over \( \mathbb{R} \) under the usual addition and scalar multiplication operations?
   - If you say that it is, then prove that each of the ten vector space conditions is satisfied.
   - If you say that it is not, then explain which vector space conditions are not satisfied.

[3] In Example 1.12, on page 84, we were introduced to the following set, along with an operation of addition and an operation of scalar multiplication:
   - Let \( \mathcal{F} = \{ f : \mathbb{R} \to \mathbb{R} \} \). That is, \( \mathcal{F} \) is the set of real-valued functions of one real variable.
   - For two functions \( f, g \) the symbol \( f + g \) denotes the function defined by \( (f + g)(x) = f(x) + g(x) \).
   - For a function \( f \) and a real number \( r \) the symbol \( r \cdot f \) denotes the function defined by \( (r \cdot f)(x) = rf(x) \).

The author states that with those operations, the set \( \mathcal{F} \) of real valued functions is a vector space, but he does not verify that any of the vector space conditions are satisfied.

Prove that the set \( \mathcal{F} \), with the operations described, does qualify to be called a vector space. That is, prove that all ten of the vector space conditions are satisfied.

Exercises using Section Two.I.2 Concepts:

[4] Let \( V \) be the vector space of \( 2 \times 2 \) matrices with real entries, with operations of matrix addition and scalar multiplication by real numbers. Let \( UT = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b \in \mathbb{R} \right\} \). This set \( UT \) is called the upper triangular \( 2 \times 2 \) matrices. The set \( UT \) is a subset of \( V \), so it inherits the operations of addition and scalar multiplication by real numbers. Is set \( UT \) a subspace? Explain clearly.

[5] The odd functions are defined as the set \( Odd = \{ f : \mathbb{R} \to \mathbb{R} \mid f(-x) = -f(x) \} \). This is a subset of the set of all real-valued functions, and so it inherits the operations of function addition and scalar multiplication by real numbers. (Those operations were defined above in problem [3].) Is the subset \( Odd \) a subspace? Explain clearly.

(Your explanation will need more detail than the book’s answer to a similar question.)