Reference 6: Finding Features in the Graph of a Rational Function $f(x)$

Part 1: Analyze the standard form of the function.

Determine the $y$-intercept by computing $f(0)$. The $y$-intercept is the point $(x, y) = (0, f(0))$.

Determine the right-end behavior by computing $\lim_{x \to \infty} f(x)$. Use the technique of identifying dominant terms.

- If $\lim_{x \to \infty} f(x) = \infty$, then the right end of the graph goes up.
- If $\lim_{x \to \infty} f(x) = -\infty$, then the right end of the graph goes down.
- If $\lim_{x \to \infty} f(x) = b$, then the line $y = b$ will be a horizontal asymptote for the right end of the graph. Because $f$ is a rational function, in this case, we know that the line $y = b$ will also be a horizontal asymptote for the left end of the graph.

If there is not a horizontal asymptote, then determine the left-end behavior by computing $\lim_{x \to -\infty} f(x)$. Use the technique of identifying dominant terms.

- If $\lim_{x \to -\infty} f(x) = \infty$, then the left end of the graph goes up.
- If $\lim_{x \to -\infty} f(x) = -\infty$, then the left end of the graph goes down.

Part 2: Analyze the factored form of the function.

Determine the effect that each factor will have on the graph of the function.

- If a factor of the form $(x - c)$ appears in the numerator only, then the graph will have an $x$-intercept at $(x, y) = (c, 0)$.
- A factor of the form $\frac{(x-c)}{(x-c)}$ will cause a hole in the graph at $x = c$. Find the $y$-coordinate of the hole by computing $\lim_{x \to c} f(x)$. The value of the limit will be the $y$-coordinate of the hole.
- A factor of the form $\frac{1}{(x-c)^2}$ will cause an asymmetrical vertical asymptote at $x = c$. Determine which side goes up and which side goes down by studying the sign behavior of $f(x)$. (Use the factored form of $f(x)$ to make a sign chart for $f(x)$.) Describe the up & down behavior at the asymptote using limit notation and the infinity symbol, $\infty$. For example:
  - If the graph goes up along the left side of the asymptote, then write $\lim_{x \to c^-} f(x) = \infty$.
  - If the graph goes down along the left side of the asymptote, write $\lim_{x \to c^-} f(x) = -\infty$.
- A factor of the form $\frac{1}{(x-c)^2}$ will cause a symmetrical vertical asymptote at $x = c$. Determine whether both sides go up or both sides go down by studying the sign behavior of $f(x)$. (Use the factored form of $f(x)$ to make a sign chart for $f(x)$.) Describe the up & down behavior at the asymptote using limit notation and the infinity symbol, $\infty$. 
