Section 5.2 The Second Derivative and Concavity

Define the second derivative

Example 1: 5 -1#54 (sim #53)

\[ f(x) = -x^4 + 50x^2 = -x^2(x^2 - 50) = -x^2(x + \sqrt{50})(x - \sqrt{50}) \]

\[ = -x^2(x + 5\sqrt{2})(x - 5\sqrt{2}) \]

\[ f'(x) = -4x^3 + 100x = -4x(x^2 - 25) = -4x(x + 5)(x - 5) \]

\[ f''(x) = -12x^2 + 100 = -12(x^2 - \frac{100}{12}) = -12\left(x^2 - \frac{25}{3}\right) \]

\[ = -12\left(x + \frac{5}{\sqrt{3}}\right)\left(x - \frac{5}{\sqrt{3}}\right) \]

Concavity

Define concavity at a particular x-value.
Define concavity on an interval
Define inflection point

Relationship between 2nd derivative and concavity

Articulate relationship between \( f'' \) and graph of \( f \)
- \( f'' \) is positive on \((a, b)\) \( \Rightarrow \) \( f \) concave up on \((a, b)\)
- \( f'' \) is negative on \((a, b)\) \( \Rightarrow \) \( f \) concave down on \((a, b)\)
- \( f'' \) is zero on \((a, b)\) \( \Rightarrow \) \( f \) not concave on \((a, b)\)

Example 1: 5 -1#54 (sim #53) \( f(x) = -x^4 + 50x^2 \)

(A) Find intervals of concavity
(B) Find x coordinates of inflection points
(C) Find y coordinates of inflection points

Quiz 12
Section 5-2 The second Derivative and concavity

The second derivative.

Symbols without variables: \( f'' \), \( D^2 f \)

Symbols with variables: \( f''(x) \), \( D^2 f(x) \), \( \frac{d^2 f(x)}{dx^2} \)

Meaning: The derivative of the derivative.

That is

\[
\frac{d^2 f(x)}{dx^2} \text{ means } \frac{d}{dx} \left( \frac{df(x)}{dx} \right)
\]
Example \( f(x) = -x^4 + 50x^2 \) From Friday to Monday

Find \( f''(x) \)

Solution

\( f'(x) = -4x^3 + 100x \)

\[
\begin{align*}
\frac{d}{dx}(f'(x)) &= f''(x) \\
&= \frac{d}{dx}(-4x^3 + 100x) \\
&= -12x^2 + 100
\end{align*}
\]
Concavity

Concavity at a particular x-value

Words: f is concave up at x = a

Meaning:
- The graph of f has a tangent line at x = a.
- For x close to a but not equal to a, the graph of f stays above that tangent line.

\[ x = 7 \]
\[ f \text{ is concave up at } x = 7 \]

\[ x = 5 \]
\[ f \text{ is concave up at } x = 5 \]
Similarly, "f is concave down at x = a" means f has a tangent line at x = a for x near a, graph of f stays below that tangent line.

Concavity on an interval

The words "f is concave up on the interval (a, b)" mean that for any particular x value a < x < b, f is concave up at that x-value.

Similarly, for concave down on an interval,
An **inflection point** is defined to be where the concavity changes from up to down or down to up.

Example

No point at $x=5$, so no inflection point.
Relationship between $f''$ and concavity of $f$.

If $f''$ is positive on interval $(a, b)$, then $f$ is concave up on $(a, b)$.
If $f''$ is negative on interval $(a, b)$, then $f$ is concave down on $(a, b)$.
If $f''$ is zero on interval $(a, b)$, then $f$ is a straight line on $(a, b)$.

Example: For $f(x) = -x^4 + 50x^2$,

(A) find intervals of concavity
(B) find $x$-coordinates of all inflection points
(C) find $y$-coordinates of all inflection points.
We found earlier that \( f''(x) = -12x^2 + 100 \)

So we should try to factor \( f''(x) \)

\[
\begin{aligned}
\sqrt{25 - 12(x^2 + x)} &= 5 - \sqrt{3}\sqrt{100 - 12(x^2 + x)} \\
&= 5 - 12(x^2 + x) \\
&= 5 - 12(x^2 + x - 25) \\
&= 5 - 12(x + 5)(x - 5) \\
&= 5 - 12(x + 5)(x + 5) \\
&= 5 - 12(x + 5)^2.
\end{aligned}
\]
Partition numbers: $x = -\frac{5}{\sqrt{3}}$  

Sign chart for $f''(x) = \begin{cases}  
-123(x + \frac{5}{\sqrt{3}})(x - \frac{5}{\sqrt{3}}) 
\end{cases}$

always negative

$-(-)(-) | -(+)(-) | -(+)(+)$

$f''$ neg  $x = -\frac{5}{\sqrt{3}}$  $f''$ pos  $x = \frac{5}{\sqrt{3}}$  $f''$ neg

$f'' = 0$  inflection  $f'' = 0$  inflection

Answers

(A) $f$ concave down on the intervals $(-\infty, -\frac{5}{\sqrt{3}})$ and $(\frac{5}{\sqrt{3}}, \infty)$

because $f''$ is negative there
(A) If concave up on the interval \((-\frac{5}{\sqrt{3}}, \frac{5}{\sqrt{3}})\)
\begin{align*}
\text{because } f''(x) &> 0
\end{align*}

(B) inflection points at \(x = -\frac{5}{\sqrt{3}}, \frac{5}{\sqrt{3}}\)

(C) \(y\)-coordinates
\begin{align*}
f(\frac{5}{\sqrt{3}}) &= -\left(\frac{5}{\sqrt{3}}\right)^4 + 50 \left(\frac{5}{\sqrt{3}}\right)^2 \\
&= -\frac{625}{81} + 50 \left(\frac{25}{9}\right) \\
&= -\frac{625}{81} + \frac{1250}{9} \\
&= -\frac{625}{81} + \frac{9(1250)}{81} \\
&= \frac{625(-1 + 18)}{81} = \frac{625(17)}{81} = \text{arrhythmia}
\end{align*}