Section 5-1 1st Derivatives and Graphs

Continuing Computational Example: For \( f(x) = -x^4 + 50x^2 \), find the intervals where the graph is increasing and decreasing.

Example 2: For \( g(x) = -xe^{(-x)} \), find the intervals where the graph is increasing and decreasing.

Local Extrema

Definition of local max

Definition of local min

Draw picture of graph that has local max & mins at smooth peaks & dips and at cusp points. Also include cusp points that have holes at the point and vertical asymptotes. Also include cubic inflection points.

Observe that at all local extrema \( f(x) \) exists and \( f'(x) = 0 \) or \( f'(x) \) DNE.

Definition of critical value

Fact: extrema only occur at critical values

notice also that local extrema all occurred at places where derivative changed sign.

First Derivative test for Local Extrema

computational examples: \( f(x) = -x^4 + 50x^2 \) and \( g(x) = -xe^{(-x)} \). In each example, find \((x, y)\) coordinates of extrema. Confirm with computer graph.

Started example \( f(x) = -x^4 + 50x^2 \).
Computational Example

\[ f(x) = -x^4 + 50x^2 \]

Find intervals where \( f(x) \) is increasing or decreasing.

Solution: Study sign behavior of \( f'(x) \)

\[ f'(x) = -4x^3 + 100x = -4x(x^2 - 25) \]
\[ = -4x(x + 5)(x - 5) \]
\[ = -4(x + 5)(x)(x - 5) \]

Partition numbers (where function = 0 or is undefined) for this are \( x = -5, x = 0, x = 5 \).
Sign chart for $f(x) = (x+5)(x)(x-5)$

- $f(x)$ is negative in the intervals $(-5, 0)$ and $(5, \infty)$.
- $f(x)$ is zero at $x = -5$, $x = 0$, and $x = 5$.
- $f(x)$ is positive in the interval $(-\infty, -5)$. 

Conclusion:
- $f(x)$ is decreasing on the intervals $(-\infty, -5)$ and $(0, 5)$.
- $f(x)$ is increasing on the interval $(5, \infty)$. 

Sign chart for $f(x) = (x+5)(x)(x-5)$
Example 2 \( g(x) = xe^{(-x)} \)

Find intervals where \( g \) is increasing or decreasing.

Solution: Find \( g' \) and study its sign behavior

\[
g'(x) = \frac{d}{dx}(xe^{(-x)})
\]

\[
= \left( \frac{d}{dx}x \right) \cdot e^{(-x)} + x \cdot \frac{d}{dx}e^{(-x)}
\]

\[
= (1) \cdot e^{(-x)} + x \cdot (-1)e^{(-x)}
\]

\[
= e^{(-x)} - xe^{(-x)}
\]

\[
= e^{(-x)} \cdot (1-x)
\]

\[
= -e^{(-x)} \cdot (x-1)
\]

\[
= \{-e^{(-x)}\} \cdot (x-1)
\]

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always negative because \( e \) (anything) is positive.
Sign chart for \( g'(x) = 3 - 5e^x - 3(x-1) \):

- Only partition numbers are \( x = 1 \).
- \( g'(x) \) changes from positive to negative at \( x = 1 \).

Conclusion:
- Increasing on the interval \((-\infty, 1)\) because \( g' \) is positive.
- Decreasing on the interval \((1, \infty)\) because \( g' \) is negative.
Local Extrema

Local maximum is a point on a graph that has the highest y-value nearby.

Local min is a point on the graph that has the lowest y-value nearby.
Definition
A critical value for a function $f$ is an $x$-value that has both of these properties:

- $f(x)$ exists
- $f'(x) = 0$ or $f'(x)$ does not exist

On our sample graph:

- $x = 5$, $x = 10$, $x = 15$, $x = 20$, $x = 35$

are all critical values.

Fact: The only place that a local max or min can occur is at a critical value.
Which critical values cause maxes?
Which ones cause mins?
Which ones are neither a max or a min?

**First Derivative Test for Local Extrema**

If \( x \) is a critical value and
\[
f'(x) \text{ changes from pos to neg at } x,
\]
then \( x \) is the location of a local max.

If \( x \) is a critical value and
\[
f'(x) \text{ changes from neg to pos at } x,
\]
then \( x \) is the location of a local min.
Return to earlier examples $f$ and $g$.

For each function

(A) Find $x$-coordinates of local extrema

(B) Find $y$-coordinates

For $f(x) = -x^4 + 50x^2$

$f'(x) = -4x^3 + 100x$

(A) Critical values $x = -5$ is min because $f'$ changes from neg to pos.

$x = 0$ is max because $f'$ changes pos to neg.

$x = 5$ is min because $f'$ changes from neg to pos.

(B) $f(-5) = -(-5)^4 + 50(-5)^2$

$= -625 + 1250$

$= 625$
\( f(0) = -(0)^4 + 50(0)^2 = 0 \)
\( f(5) = -(5)^4 + 50(5)^2 = 625 \)

\( \max \quad (-5, 625) \)
\( \min \quad (0, 0) \)
\( \min \quad (5, 625) \)