<table>
<thead>
<tr>
<th>Section 4-6 Related Rates</th>
</tr>
</thead>
</table>

**Terminology of Rates**
- Remember rate of change of \( f(t) \) is \( f'(t) \)
- Other variables, for example \( A \) \( A(t) \) a function of the variable \( t \)
- \( \frac{d}{dt}A(t) \) = rate of change of \( A \) with respect to \( t \)

**Idea of Related Rates**
- Related rates: start with an equation involving a bunch of letters some of which depend on time. This equation expresses a relationship among the letters. Take derivative of both sides of the equation with respect to \( t \). Result will be a new equation involving the letters and their time derivatives (also called rates of change). This new equation expresses a relationship between all these things. Thus the term “related rates”

**Examples involving triangles**

**Example #1**
(similar to Section 4-6 Example 1 and suggested problem 4-6#17)
- 20 foot ladder placed against a vertical wall.
- Bottom of ladder is moving away from wall at 1 ft/sec.
- How fast is the top sliding down the wall at the instant when the bottom of the ladder is 12 feet from wall?

**Example #2**
(similar to Section 4-6 Example 2 and suggested problem 4-6#10)
- An airliner is flying at a constant speed of 550 mph at an altitude of 6 miles. The plane flies over a spot on the ground that is 8 miles from the control tower.
- (A) How far is the plane from the control tower at that instant?
- (B) How fast is the distance from the plane to the control tower decreasing at that instant?
Monday February 21, 2011 (Day 28)

We will omit Section 4-5

Implicit Differentiation

Bring your O.U. Student I.D. to Thursday's Exam
Section 4-6 Related Rates

Terminology of Rates

If $f$ is a function and "$a$" is some number, then the symbol $f'(a)$ is called the "instantaneous rate of change of $f$ at $a$".

In general, the "rate of change of $f$ at some particular time," you are referring to the quantity $f'(a)$ at the instant in time the derivative of $f$. 
We get a little sloppy about this. For a function \( f(t) \), we sometimes refer to the derivative of \( f \) as \( f'(t) \) as the rate of change of \( f \).

The idea of "Related Rates" (the subject of 4-6)

Start with an equation involving a bunch of letters; some of which represent functions of time and some of which represent constants. This equation expresses a relationship among the letters. Left side = right side
Take the derivative of both sides of this equation:

\[ \frac{d}{dt} (\text{left side}) = \frac{d}{dt} (\text{right side}) \]

This will probably involve the chain rule. The result will be a new equation involving the original letters and their rates of change. The equation expresses a relationship among the rates of change. Hence the term "related rates."
Examples involving Triangles

Example #1 (similar to book example and suggested problem)

A 20-foot ladder is placed against a vertical wall.

The bottom of the ladder is moving away from the wall at a rate of 1 foot/second.

How fast is the top of the ladder sliding down the wall at the instant when the bottom of the ladder is 12 feet from the wall?

\[
\frac{da}{dt} = a' = 1 \text{ foot/second}
\]

\[
b = ?
\]

\[
b' = ?
\]

We need to find this.
In this problem
letters are $a$, $b$, $c$
$a$ and $b$ are functions of time $a(t), b(t)$
$c$ is a constant

Equation relating $a$, $b$, $c$

\[ a^2 + b^2 = c^2 \]  Pythagorean Theorem

Make the time dependence explicit

\[ (a(t))^2 + (b(t))^2 = c^2 \]

Take the derivative of both sides of this equation.
\( \frac{d}{dt} \left( \frac{(ae(t))^2 + (be(t))^2}{2} \right) = c^2 \text{ constant} \)

\( \frac{d}{dt} \left( \frac{(ae(t))^2 + (be(t))^2}{2} \right) = 0 \)

\( \frac{d}{dt} \left( \frac{(ae(t))^2 + (be(t))^2}{2} \right) = \) nested function

\( \frac{d}{dt} \left( \frac{(ae(t))^2 + (be(t))^2}{2} \right) = \) nested function

We will need to use the chain rule

Chain Rule: For \( \frac{d}{dt} \left( \frac{(ae(t))^2 + (be(t))^2}{2} \right) = c \),

inner: \( \frac{d}{dt} \left( \frac{(ae(t))^2}{2} \right) = \frac{a'e(t)}{2} \)

outer: \( \frac{d}{dt} \left( \frac{be(t)}{2} \right) = \frac{b'e(t)}{2} \)

outer' \( = 2 \)
Assemble the derivatives

\[
\frac{d}{dt} \text{outer}(\text{inner}(t)) = \text{Outer}'(\text{inner}(t)) \cdot \text{inner}'(t)
\]

\[
2(a(t)) \cdot a'(t) + 2(b(t)) \cdot b'(t) = 0
\]

Clean up
\[
2 \cdot a(t) \cdot a'(t) + 2 \cdot b(t) \cdot b'(t) = 0
\]

Clean up some more by hiding the time dependence
\[
2 \cdot a \cdot a' + 2 \cdot b \cdot b' = 0
\]
We need to solve this equation for $b'$:

$$2 \cdot b \cdot b' = -2 \cdot a \cdot a'$$

$$b' = -\frac{2a a'}{2b}$$

$$b' = -\frac{a a'}{b}$$

So if we can substitute in $a$, $a'$, $b$, we will have $b'$.

We need to find $b$. Use the original equation $a^2 + b^2 = c^2$.

$$b^2 = c^2 - a^2$$

$$b = \sqrt{c^2 - a^2}$$
Substitute in $c = 20$ and $a = 12$

$$b = \sqrt{20^2 - 12^2}$$

$$= \sqrt{400 - 144}$$

$$= \sqrt{256}$$

$$= 16 \text{ feet}$$
So

\[ b' = - \frac{a \cdot a'}{b} \]

\[ = - \frac{12 \text{ feet} - 1 \text{ foot}}{16 \text{ feet}} \text{ second} \]

\[ b' = - \frac{3}{4} \text{ feet per second} \]

Conclude that top of the ladder is sliding down the wall at a rate of \( \frac{3}{4} \) foot per second.
Example #2

An airliner is flying at a constant speed of 550 mph at an altitude of 6 miles. The plane flies over a spot on the ground that is 8 miles from the control tower. At that instant, how far is the plane from the control tower? At that instant, how fast is the distance from the plane to the tower decreasing?
Solution

\[ a^2 + b^2 = c^2 \]

with time dependence

\[ (a(t))^2 + b^2 = (c(t))^2 \]

take the derivative of both sides

\[ \frac{d}{dt}((a(t))^2 + b^2) = \frac{d}{dt}(c(t))^2 \]
\[ \frac{d}{dt}(a(t)^2) + \frac{d}{dt}b^2 = \frac{d}{dt}(c(t)^2) \]

- Chain rule
- Constant
- Chain rule

\[ 2 \cdot a(t) \cdot a'(t) + 0 = 2 \cdot c(t) \cdot c'(t) \]

Clean up

\[ 2 \cdot a \cdot a' = 2 \cdot c \cdot c' \]

Now answer the questions

(A) value of c?

\[ a^2 + b^2 = c^2 \]

\[ c = \sqrt{a^2 + b^2} \]
\[ C = \sqrt{(8 \text{ miles})^2 + (6 \text{ miles})^2} \]

\[ = \sqrt{64 + 36} \text{ miles} \]

\[ = \sqrt{100} \text{ miles} \]

\[ = 10 \text{ miles} \]

(B) To get \( c \), we solve the equation for \( c \).

\[ 2a + 1 = 2cc \]

The result is \( c = \frac{a}{a} \).

\[ c = \frac{a}{a} \]
Substitute in values for \( a, a_1, c \), we get

\[ C' = \frac{8 \text{ miles} \cdot (550 \text{ miles per hour})}{10 \text{ miles}} \]

\[ C' = -440 \text{ miles per hour} \]