Section 4-1
Review Tuesday discussion of Interest of Bank Accounts

Simple Interest Formula: \( A = P + Prt = P(1 + rt) \).

Periodically Compounded Interest Formula: \( A = P \left(1 + \frac{r}{m}\right)^{mt} \).

We found that \( \lim_{m \to \infty} A = \lim_{m \to \infty} P \left(1 + \frac{r}{m}\right)^{mt} = e^{rt} \).

Inspired by this, we can invent a bank account that uses the formula \( A = Pe^{rt} \) to compute its balance. Call it "continuously compounded interest".

Problems involving Continuously Compounded Interest

Underlying idea: solve the equation \( A = Pe^{rt} \) for each letter in terms of the others. (do this)

Example 1: Deposit $885 into account with 4.3% interest compounded continuously. What is the balance after 10 years?

Solution: \( A = Pe^{rt} = 885e^{(0.043)(10)} \approx $1360.47 \).

Example 2: Deposit $885 into account with 4.3% interest compounded continuously. How long after the initial deposit until the initial amount has grown to $1500?

Solution: \( t = \frac{\ln(1500)}{0.043} \approx 12.27 \text{ years} \).

Example 3: Deposit some money into an account earning 5% interest compounded continually. How long after the initial deposit until the initial amount has tripled?

Solution: \( t = \frac{\ln(3)}{0.05} \approx 21.97 \text{ years} \).

Section 4-2 Derivatives of Exponential Functions

three new derivative rules:

\[
\frac{d}{dx} e^{x} = e^{x}
\]
\[
\frac{d}{dx} b^{x} = b^{x} \cdot \ln(b)
\]
\[
\frac{d}{dx} e^{cx} = c \cdot e^{cx}
\]

Example: Find the derivatives of the following functions:

(A) \( f(x) = 11e^{x} \)
(B) \( f(x) = 11x^{e} \)
(C) \( f(x) = 11e^{12} \)
(D) \( f(x) = 11 \cdot 12^{x} \)
(E) \( f(x) = 11e^{(12x)} \)
Review Tuesday: Interest of Bank Accounts

Simple Interest

\[ A = P + Prt \]
\[ = P(1 + rt) \]

Periodically Compounded Interest

\[ A = P \left(1 + \frac{r}{m}\right)^{mt} \]

- Each time the interest gets compounded, the slope gets steeper.
- Graph of actual balance, not the formula.
Think of the equation $A = P \left(1 + \frac{r}{n}\right)^{nt}$ as having $M$ as its variable $A_m = P \left(1 + \frac{r}{m}\right)^{mt}$.

We asked: what happens when interest is compounded more frequently? That is, what is the limit

$$\lim_{m \to \infty} A_m = \lim_{m \to \infty} P \left(1 + \frac{r}{m}\right)^{mt}$$
We studied \( \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \)

It seemed that the limit existed and was a number near 2.718.

We introduced a symbol \(e\) for the real number that is the value of that limit.

That is \( e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \)

Fact \( e \approx 2.718 \) but \( e \) is irrational.

**Related Limits**

1. \( \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^{(x)} \)
2. \( \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^{nx} = e^{(x^2)} \)
(3) \( \lim_{m \to \infty} P \left( 1 + \frac{r}{m} \right)^{mt} = P e^{rt} \) 

So we see that

\[ \lim_{m \to \infty} A_m = \lim_{m \to \infty} P \left( 1 + \frac{r}{m} \right)^{mt} = P e^{rt} \]

Inspired by this, just invent a new formula for bank account interest:

\[ A = P e^{rt} \]

"Continuously-compounded interest"
Examples of Problems Involving Continuously-compounded interest.

Equation \( A = Pe^{rt} \) involves \( A, P, r, t \) and is solved for \( A \).

Solve the equation for \( P \) by dividing both sides by \( e^{rt} \).

Result: \( P = \frac{A}{e^{rt}} \) solved for \( P \).

Solve the original equation for \( t \)

\[
A = Pe^{rt} \\
\frac{A}{P} = e^{rt} \\
ln\left(\frac{A}{P}\right) = ln(e^{rt}) \\
ln\left(\frac{A}{P}\right) = rt \\
t = \frac{ln\left(\frac{A}{P}\right)}{r} \text{ solved for } t
\]
To solve for $r$, go back to step 2.

\[
\ln \left( \frac{A}{P} \right) = rt
\]

Solve for $r$.

Example: Deposit $885 into account with 4.3% interest compounded continuously. What is the balance after 10 years?

Solution: $P = 885$, $r = 0.043$, $t = 10$, $A =$ unknown

Substitute values in.

\[
A = 885 \cdot e^{(0.043)(10)} 
\]

\[
A \approx 1360.47
\]
Example 2

Deposit $885 into account with 4.3% interest compounded continuously.
How long until the balance is $1500?

Solution

\[ P = 885 \]
\[ r = 0.043 \]
\[ A = 1500 \]
\[ t = \text{unknown} \]

Substitute into formula for \( t \)

\[ t = \frac{\ln(A/P)}{r} = \frac{\ln(1500/885)}{0.043} \approx 12.27 \text{ years} \]
Example #3

Deposit some money into an account earning 5% interest compounded continuously. How long until the balance triples?

Solution

\[ P = \text{unknown} \]
\[ r = 0.05 \]

We need to find \( t \) \( \rightarrow \) \( t = \text{unknown} \)

\( A = 3P \) balance is 3 times the initial balance.

Substitute into formula for \( t \)

\[ t = \frac{\ln(\frac{A}{P})}{r} = \frac{\ln(\frac{3P}{P})}{0.05} = \frac{\ln(3)}{0.05} \approx 21.97 \text{ years} \]
Section 4-2 Derivatives of Exponential Functions

New derivative rules

- If $f(x) = e^{x}$ then $f'(x) = e^{x}$
  
  Single equation version: $\frac{d}{dx} e^{x} = e^{x}$

- If $f(x) = b^{x}$ then $f'(x) = b^{x} \ln(b)$
  
  Single equation version: $\frac{d}{dx} b^{x} = b^{x} \cdot \ln(b)$

- If $f(x) = e^{cx}$ then $f'(x) = c \cdot e^{cx}$
  
  Single equation version: $\frac{d}{dx} e^{cx} = c \cdot e^{cx}$
Examples

Find these derivatives.
(A) \( f(x) = 11e^x \)
(B) \( f(x) = 11xe^x \)
(C) \( f(x) = 11e^{x^2} \)
(D) \( f(x) = 11e^{x^2 + 1} \)
(E) \( f(x) = 11e^{-x} \)

\[
\begin{align*}
\frac{d}{dx}e^x & = e^x \\
\frac{d}{dx}xe^x & = xe^x + e^x \\
\frac{d}{dx}e^{x^2} & = 2xe^{x^2} \\
\frac{d}{dx}e^{x^2+1} & = (2x)e^{x^2+1} \\
\frac{d}{dx}e^{-x} & = -e^{-x}
\end{align*}
\]

\( \text{constant} \)
\( \text{exponential} \)
\( \text{power} \)