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Drill for Section 1.3.1: Theorems about Axiom System #1

In Section 1.3.1, you saw the following axiom system.

<table>
<thead>
<tr>
<th>Axiom System:</th>
<th>Axiom System #1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Primitive Relations:</td>
<td>relation on the set $\mathbb{Z}$, spoken “$x$ is related to $y$”</td>
</tr>
</tbody>
</table>
| Axioms: | (1) $5$ is related to $7$  
         | (2) $5$ is related to $8$  
         | (3) For all integers $x$ and $y$, if $x$ is related to $y$, then $y$ is related to $x$.  
         | (4) For all integers $x$, $y$, and $z$, if $x$ is related to $y$ and $y$ is related to $z$, then $x$ is related to $z$. |

With the symbols and terminology of relations from Math 306, the presentation of the axiom system was abbreviated as follows.

<table>
<thead>
<tr>
<th>Axiom System:</th>
<th>Axiom System #1, abbreviated version</th>
</tr>
</thead>
<tbody>
<tr>
<td>Primitive Relations:</td>
<td>relation $\mathcal{R}$ on the set $\mathbb{Z}$, spoken “$x$ is related to $y$”</td>
</tr>
</tbody>
</table>
| Axioms: | (1) $5\,\mathcal{R}\,7$  
         | (2) $5\,\mathcal{R}\,8$  
         | (3) relation $\mathcal{R}$ is symmetric  
         | (4) relation $\mathcal{R}$ is transitive |

In the text, you saw the following theorem and proof:

**Theorem for Axiom System #1: $7$ is related to $8$.**

**Proof**

1. $7$ is related to $5$ (by axioms (1) and (3))
2. $7$ is related to $8$ (by statement (1) and axioms (2) and (4))

End of proof

The first goal is to prove the following four theorems:

**Group A:** Theorem for Axiom System #1: $8$ is related to $7$.

**Group B:** Theorem for Axiom System #1: $5$ is related to itself

**Group C:** Theorem for Axiom System #1: $7$ is related to itself

**Group D:** Theorem for Axiom System #1: $8$ is related to itself

For discussion: In light of the theorems proven by groups $B$, $C$, and $D$, would you say that the relation is reflexive?
Drills for Section 2.2: Justifying Steps in Proofs about Five-Point Geometry

<table>
<thead>
<tr>
<th>Axiom System:</th>
<th>Five-Point Geometry (introduced in class)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Primitive Objects:</td>
<td>point, line</td>
</tr>
<tr>
<td>Primitive Relation:</td>
<td>the point lies on the line</td>
</tr>
<tr>
<td>Axioms:</td>
<td>&lt;1&gt; There are five points. These may be denoted ( P_1, P_2, P_3, P_4, P_5 ).</td>
</tr>
<tr>
<td></td>
<td>&lt;2&gt; For any two distinct points, there is exactly one line that both points lie on.</td>
</tr>
<tr>
<td></td>
<td>&lt;3&gt; For any line, there are exactly two points that lie on the line.</td>
</tr>
</tbody>
</table>

Justify the steps in the following two theorems.

Five-Point Geometry Theorem #1: There are exactly ten lines.

**Proof**

**Part 1: Show that ten lines exist**

1. There must be a line that both points \( P_1 \) and \( P_2 \) lie on. (justify) Call it \( L_1 \).
2. There must be a line that both points \( P_1 \) and \( P_3 \) lie on. (justify)
3. The line that points \( P_1 \) and \( P_3 \) lie on cannot be \( L_1 \). (justify) So it must be a new line. Call it \( L_2 \).
4. We see that for each pair of points, there must be a line that both points, and no other points, lie on. Ten unique pairs of points can be made from the five points \( P_1, P_2, P_3, P_4, P_5 \). So there must be at least ten lines. Call them \( L_1, L_2, \ldots, L_{10} \).

**Part 2: Show that there cannot be more than ten lines**

5. Suppose that there is an eleventh line. (justify) Call it \( L_{11} \).
6. Line \( L_{11} \) must pass through two points. (justify) Call the two points \( A \) and \( B \). So points \( A \) and \( B \) lie on line \( L_{11} \).
7. Points \( A \) and \( B \) must be two of the five points \( P_1, P_2, P_3, P_4, P_5 \). So points \( A \) and \( B \) also lie on one of the lines from the list \( L_1, L_2, \ldots, L_{10} \).
8. We have reached a contradiction. (explain the contradiction) Therefore, our assumption in step (5) was wrong. There cannot be an eleventh line.

**End of Proof**
Five-Point Geometry Theorem #2: Every point lies on exactly four lines.

Proof

(1) Let $P$ be any point. We can rename the five points so that $P$ is $P_1$.

Part 1: Show that $P$ lies on at least four lines.

(2) There must be a line that passes through $P$ and $P_2$. Call it $L_1$.
(3) There must be a line that passes through $P$ and $P_3$. (justify)
(4) The line that passes through $P$ and $P_3$ cannot be $L_1$. (justify) So it must be a new line. Call it $L_2$.
(5) There must be a line that passes through $P$ and $P_4$. (justify)
(6) The line that passes through $P$ and $P_4$ cannot be $L_1$ or $L_2$. (justify) So it must be a new line. Call it $L_3$.
(7) There must be a line that passes through $P$ and $P_5$. (justify)
(8) The line that passes through $P$ and $P_5$ cannot be $L_1$ or $L_2$ or $L_3$. (justify) So it must be a new line. Call it $L_4$.

Part 2: Show that $P_1$ cannot lie on a fifth line.

(9) Suppose that $P$ lies on a fifth line. (justify) Call it $L_5$.
(10) Line $L_5$ must pass through another point. (justify) Call the second point $Q$. So points $P$ and $Q$ lie on line $L_5$.
(11) Point $Q$ must be one of the four points $P_2, P_3, P_4, P_5$. So points $P$ and $Q$ also lie on one of the lines $L_1, L_2, L_3, L_4$.
(12) We have reached a contradiction. (explain the contradiction) Therefore, our assumption in step (9) was wrong. $P$ cannot lie on a fifth line.

End of Proof
Drill for Section 2.5: Justifying Fano’s Geometry Theorem 1

<table>
<thead>
<tr>
<th>Axiom System:</th>
<th>Fano’s Geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>Primitive Objects:</td>
<td>point, line</td>
</tr>
<tr>
<td>Primitive Relation:</td>
<td>The point lies on the line.</td>
</tr>
<tr>
<td>Axioms:</td>
<td>&lt;F1&gt; There exists at least one line.</td>
</tr>
<tr>
<td></td>
<td>&lt;F2&gt; Exactly three points lie on each line.</td>
</tr>
<tr>
<td></td>
<td>&lt;F3&gt; There does not exist a line that all the points of the geometry lie on.</td>
</tr>
<tr>
<td></td>
<td>&lt;F4&gt; For any two points, there is exactly one line that both points lie on.</td>
</tr>
<tr>
<td></td>
<td>&lt;F5&gt; For any two lines, there is at least one point that lies on both lines.</td>
</tr>
</tbody>
</table>

Fano’s Geometry Theorem #3: There exist exactly seven points.

Justify the steps in the following Proof of Fano’s Theorem #3

Part 1: Show that there must be at least seven points.

Introduce Line $L_1$ and points $A, B, C, D$.
1. There exists a line. (justify) We can call it $L_1$.
2. There are exactly three points on $L_1$. (justify) We can call them $A, B, C$.
3. There must be a point that does not lie on $L_1$. (justify) We can call it $D$.

Introduce Line $L_2$ and point $E$.
4. There must be a line that both $A$ and $D$ lie on. (justify)
5. The line that both $A$ and $D$ lie on cannot be $L_1$. (justify) So it must be a new line. We can call it $L_2$.
6. There must be a third point that lies on $L_2$. (justify)
7. The third point on $L_2$ cannot be $B$ or $C$. (justify) So it must be a new point. We can call it $E$.

Introduce Line $L_3$ and point $F$.
8. There must be a line that both $B$ and $D$ lie on. (justify)
9. The line that both $B$ and $D$ lie on cannot be $L_1$ or $L_2$. (justify) So it must be a new line. We can call it $L_3$.
10. There must be a third point that lies on $L_3$. (justify)
11. The third point on $L_3$ cannot be $A, C, or E$. (justify) So it must be a new point. We can call it $F$.

The proof continues on back ➔
**Introduce Line \( L_4 \) and point \( G \).**

(12) There must be a line that both \( C \) and \( D \) lie on. (justify)
(13) The line that both \( C \) and \( D \) lie on cannot be \( L_1 \) or \( L_2 \) or \( L_3 \). (justify) So it must be a new line. We can call it \( L_4 \).
(14) There must be a third point that lies on \( L_4 \). (justify)
(15) The third point on \( L_3 \) cannot be \( A, B, C, E, \) or \( F \). (justify) So it must be a new point. We can call it \( G \).

**Part 2: Show that there cannot be an eight point.**

(16) Suppose there is an eighth point. (justify) Call it \( H \).
(17) There must be a line that both \( A \) and \( H \) lie on. (justify)
(18) The line that both \( A \) and \( H \) lie on cannot be \( L_1 \) or \( L_2 \) or \( L_3 \) or \( L_4 \). (justify) So it must be a new line. We can call it \( L_5 \).
(19) There must be a third point that lies on \( L_5 \). (justify)
(20) Line \( L_5 \) must intersect each of the lines \( L_1 \) and \( L_2 \) and \( L_3 \) and \( L_4 \). (justify)
(21) The third point on \( L_5 \) must be \( G \). (justify. Be sure to explain clearly)
(22) So points \( A, G, H \) lie on \( L_5 \).
(23) We have reached a contradiction. (explain the contradiction) Therefore, our assumption in step (16) was wrong. There cannot be an eighth point.

**End of proof**
Drill for Section 2.7: Theorems of Incidence Geometry

<table>
<thead>
<tr>
<th>Axiom System:</th>
<th>Incidence Geometry (Introduced in Section 2.7.2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Primitive Objects:</td>
<td>point, line</td>
</tr>
<tr>
<td>Primitive Relations:</td>
<td>the point lies on the line</td>
</tr>
<tr>
<td>Axioms:</td>
<td>$&lt;I1&gt;$ There exist three distinct non-collinear points. (at least three) $&lt;I2&gt;$ For every pair of distinct points, there exists exactly one line that both points lie on. $&lt;I3&gt;$ At least two points lie on every line.</td>
</tr>
</tbody>
</table>

In this class drill, you will study the proofs of five Incidence Geometry theorems.

**Theorem 1:** In Incidence Geometry, if $L$ and $M$ are distinct lines that are not parallel, then there is exactly one point that both lines pass through.

**Justify the steps in the following proof of Theorem 1.**

1. Suppose that lines $L$ and $M$ are distinct, non-parallel lines.
2. Lines $L$ and $M$ intersect. *(justify)*
3. There must be at least one point that both lines $L$ and $M$ pass through. *(justify)* We can call one such point $P$.
4. Assume that there is more than one point that both lines pass through. *(justify)* Then there is a second point, that we can call $Q$.
5. There are two lines that pass through points $P$ and $Q$. *(justify)*
6. We have reached a contradiction. *(explain the contradiction)* So our assumption in (4) was wrong. There cannot be more than one point that both lines pass through.
7. So there must be exactly one point that both lines $L$ and $M$ pass through.

**End of Proof**

Theorem 2: In Incidence Geometry, there exist three lines that are not concurrent

**Justify the steps in the following proof of Theorem 2.**

1. Suppose that it is not true that there exist three lines that are not concurrent. *(justify)*
2. Any set of three lines is concurrent. *(justify)*
3. Given any set of three lines, there exists at least one point that all three lines pass through. *(justify)*
4. There exist three non-collinear points. *(justify)* We can call them $A$, $B$, $C$.
5. There exists a line that passes through points $A$ and $B$. *(justify)* We can denote this line by the symbol $\overline{AB}$.
6. Line $\overline{AB}$ does not pass through point $C$. *(justify)*
7. Similarly, there exists a line $\overline{BC}$ that passes through $B$ and $C$ and does not pass through $A$, and a line $\overline{CA}$ that passes through $C$ and $A$ and does not pass through $B$. 
(8) There exists at least one point that all three lines $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$ pass through. \textbf{(justify)}

(9) Any point that all three lines $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$ pass through cannot be point $A, B$, or $C$ \textbf{(justify)}, so it must be a new point that we can call point $D$.

(10) There are two lines that pass through points $A$ and $D$. \textbf{(justify)}

(11) We have reached a contradiction. \textbf{(explain the contradiction)} So our assumption in (1) was wrong. It must be true that there exist three lines that are non-concurrent.

\textbf{End of Proof}

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\textbf{Preparation for Proof of Theorems 3, 4, and 5: Writing negations of statements}

Call this statement $S$: “Given any line $L$, there exists a point not lying on $L$.”

**Write the negation, $\sim S$:**

Call this statement $T$: “Given any point $P$, there exists a line that does not pass through $P$.”

**Write the negation, $\sim T$:**

Call this statement $U$: “Given any point $P$, there exist two lines that pass through $P$.”

**Write the negation, $\sim U$:**

-------------------------------------------------------------------------------------------------------------

\textbf{Theorem 3: In Incidence Geometry, given any line $L$, there exists a point not lying on $L$.}

\textbf{Justify the steps in the following proof of Theorem 3.}

(1) Suppose that it is not true that given any line $L$, there exists a point not lying on $L$. \textbf{(justify)}

(2) Every point lies on $L$. \textbf{(justify)}

(3) The set of all the points in the geometry is collinear. \textbf{(justify)}

(4) We have reached a contradiction. \textbf{(explain the contradiction)} So our assumption in (1) was wrong. It must be true that given any line $L$, there exists a point not lying on $L$.

\textbf{End of Proof}

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\textbf{Theorem 4: In Incidence Geometry, given any point $P$, there exists a line that does not pass through $P$.}

Create a proof of Theorem 4 that uses the method of contradiction.

-------------------------------------------------------------------------------------------------------------

\textbf{Theorem 5: In Incidence Geometry, given any point $P$, there exist two lines that pass through $P$.}

Try to create a proof of Theorem 5. This one will be harder.
Drill for Section 4.1: Introduction to The Separation Axiom and Half-Planes

Justify the steps in the proof of the following theorem. Draw a picture to illustrate.

**Theorem 21**: Each half-plane contains at least three non-collinear points.

**Proof**

1. Given any line, call it $L_1$.

**Part I: Introduce Half-Plane $H_1$ and show that it contains three non-collinear points.**

2. There exists a point not on $L_1$. (justify) Call it $P_1$.
3. $P_1$ lies in one of the two half-planes determined by line $L_1$. (justify) Call it $H_1$.

**Introduce line $L_2$.**

4. There exist two distinct points on $L_1$. (justify) Call them $P_2$ and $P_3$.
5. There exists a unique line passing through $P_1$ and $P_2$. (justify)
6. The line passing through $P_1$ and $P_2$ is not $L_1$. (justify) So it must be new. Call it $L_2$.

**Introduce line $L_3$.**

7. There exists a unique line passing through $P_1$ and $P_3$. (justify)
8. The line passing through $P_1$ and $P_3$ is not $L_1$ or $L_2$. (justify) So it must new. Call it $L_3$.

**Introduce point $P_4$.**

9. There exists a point such that $P_2 \neq P_1 \neq P_3$. (justify)
10. This point cannot be the same as any of our previous three points. (justify) So it must be a new point. Call it $P_4$. So $P_2 \neq P_1 \neq P_4$.
11. $P_4$ is in half-plane $H_1$. (justify)

**Introduce point $P_5$.**

12. There exists a point such that $P_3 \neq P_1 \neq P_2$. (justify)
13. This must be a new point. (justify) Call it $P_5$. So $P_3 \neq P_1 \neq P_5$.
14. $P_5$ is in half-plane $H_1$. (justify)

**Conclusion of Part I:**

15. Points $P_1$ and $P_4$ and $P_5$ are non-collinear. (justify)

**Part II: Show that Half-Plane $H_2$ contains three non-collinear points.**

**Introduce point $P_6$.**

16. There exists a point such that $P_1 \neq P_2 \neq P_3$. (justify)
17. This must be a new point. (justify) Call it $P_6$. So $P_1 \neq P_2 \neq P_6$.
18. $P_6$ is in half-plane $H_2$. (justify)

**Introduce point $P_7$.**

19. There exists a point such that $P_2 \neq P_6 \neq P_3$. (justify)
20. This must be a new point. (justify) Call it $P_7$. So $P_2 \neq P_6 \neq P_7$.
21. $P_7$ is in half-plane $H_2$. (justify)

**Introduce point $P_8$.**

22. There exists a point such that $P_1 \neq P_3 \neq P_6$. (justify)
23. This must be a new point. (justify) Call it $P_8$. So $P_1 \neq P_3 \neq P_8$.
24. $P_8$ is in half-plane $H_2$. (justify)

**Conclusion of Part II:**

25. Points $P_6$ and $P_7$ and $P_8$ are non-collinear. (justify)

**End of Proof**
Drill for Section 4.2: Theorems about lines intersecting triangles

Justify the steps in the proof of the following theorem. Draw a picture to illustrate.

Theorem: (Pasch’s Theorem) about a line intersecting a side of a triangle between vertices
If a line intersects the side of a triangle at a point between vertices, then the line also intersects at least one of the other two sides.

Proof
(1) Suppose that line \( L \) intersects side \( \overline{AB} \) of \( \triangle ABC \) at a point \( D \) such that \( A \neq D \neq B \).
(2) Points \( A \) and \( B \) are on opposite sides of line \( L \). (justify) Let \( H_A \) and \( H_B \) be their respective half-planes.
(3) Exactly one of the following statements is true. (justify)
   (i) \( C \) lies on \( L \).
   (ii) \( C \) is in \( H_A \).
   (iii) \( C \) is in \( H_B \).
Case (i)
(4) If \( C \) lies on \( L \), then \( L \) intersects both \( \overline{AC} \) and \( \overline{BC} \) at point \( C \). (justify)
Case (ii)
(5) If \( C \) is in \( H_A \), then points \( B \) and \( C \) are lie on opposite sides of \( L \). (justify)
(6) In this case, \( L \) will intersect \( \overline{BC} \) at a point between \( B \) and \( C \). (justify)
Case (iii)
(7) If \( C \) is in \( H_B \), then points \( A \) and \( C \) are lie on opposite sides of \( L \). (justify)
(8) In this case, \( L \) will intersect \( \overline{AC} \) at a point between \( A \) and \( C \). (justify)
Conclusion of cases
(9) In every case, we see that \( L \) intersects \( \overline{AC} \) or \( \overline{BC} \) or both.

End of Proof
Drill for Section 4.4: Theorems about rays and lines intersecting triangle interiors

The goal of Section 4.4 of the textbook is for you to appreciate the difficulty and have some understanding of the structure of the proof of Theorem 32, The Crossbar Theorem. The first seven theorems of this section of the book (Theorems 25 – 31) are used in proving Theorem 32. Proving any of those seven theorems, or proving Theorem 32, is a daunting task, and it is very easy to get stuck in the details and never finish. To not get stuck in the details at the beginning and to get some understanding of the big picture, one should … make little pictures.

Each of the seven groups should make a chalkboard picture that illustrates the statement of one of the following seven preliminary theorems. (Don’t try to illustrate the proof, just the theorem statement.) Then we will discuss the proof of Theorem 32 as a group.

**Theorem 25:** about a ray with an endpoint on a line
   If a ray that has its endpoint on a line but does not lie in the line, then all points of the ray except the endpoint are on the same side of the line.

**Theorem 26:** Corollary about a segment that has an endpoint on a line
   If a segment that has an endpoint on a line but does not lie in the line, then all points of the segment except that endpoint are on the same side of the line.

**Theorem 27:** Points on a side of a triangle are in the interior of the opposite angle.
   If a point lies on the side of a triangle and is not one of the endpoints of that side, then the point is in the interior of the opposite angle.

**Theorem 28:** about a ray with its endpoint on an angle vertex
   If a ray has its endpoint on an angle vertex and passes through a point in the angle interior, then every point of the ray except the endpoint lies in the angle interior.

**Theorem 29:** Lemma about a point on a ray whose endpoint is an angle vertex
   Given angle $\angle ABC$ and points $D, E, F$ that satisfy $B \ast A \ast D$ and $A \ast E \ast C$ and $B \ast E \ast F$, the point $F$ will lie in the interior of $\angle DAC$.

**Theorem 30:** Lemma about a segment and a ray with endpoints on a line
   If points $A$ and $B$ lie on opposite sides of line $\overline{CD}$, then segment $\overline{CA}$ does not intersect ray $\overline{DB}$.

**Theorem 31:** a more specific version of Pasch’s Theorem about a line intersecting the side of a triangle at a point between two vertices
   If a line $L$ intersects side $\overline{AB}$ of $\triangle ABC$ at a point $D$ such that $A \ast D \ast B$ and line $L$ also passes through a point $E$ that is on the same side of line $\overline{AB}$ as point $C$, then ray $\overline{DE}$ also intersects the triangle at a point that lies on at least one of the other two sides.
Drill for Section 5.3: Existence and Uniqueness of the Angle Bisector

Justify the steps in the proof of the following theorem. Draw a picture to illustrate. (Your justifications may refer to any prior theorem and to Neutral Axioms <N1> through <N8>. You may not use Axioms <N9> or <N10>.)

Theorem 36: Every angle has a unique bisector.

Proof

(1) Suppose that angle $\angle ABC$ is given.

Part 1: Introduce special ray $\overrightarrow{BD}$.

(2) Let $r = \frac{1}{2}m(\angle ABC)$. Let $H_C$ be the half-plane created by line $\overrightarrow{AB}$ that contains point $C$.

Observe that ray $\overrightarrow{BA}$ lies on the edge of this half-plane.

(3) There exists a ray $\overrightarrow{BD}$ such that $D \in H_C$ and $m(\angle ABD) = r$. (justify)

Part 2: Show that points $D$ and $A$ lie on the same side of line $\overrightarrow{BC}$.

(4) There are three possibilities for point $D$. (justify)

(i) Point $D$ lies on line $\overrightarrow{BC}$.

(ii) Points $D$ and $A$ lie on opposite sides of line $\overrightarrow{BC}$.

(ii) Points $D$ and $A$ lie on the same side of line $\overrightarrow{BC}$.

Case (i): Point $D$ lies on line $\overrightarrow{BC}$.

(5) Suppose that point $D$ lies on line $\overrightarrow{BC}$. (assumption)

(6) Then ray $\overrightarrow{BD}$ is the same ray as $\overrightarrow{BC}$. (justify) So angle $\angle ABD$ is the same angle as $\angle ABC$.

(7) This contradicts that $m(\angle ABD) = r = \frac{1}{2}m(\angle ABD)$. Therefore our assumption in step (5) was wrong. Point $D$ cannot lie on line $\overrightarrow{BC}$.

Case (ii): Points $D$ and $A$ lie on opposite sides of line $\overrightarrow{BC}$.

(8) Suppose that points $D$ and $A$ lie on opposite sides of line $\overrightarrow{BC}$.

(9) Segment $\overrightarrow{AD}$ intersects line $\overrightarrow{BC}$ at a point $E$ between $A$ and $D$. (justify)

(10) Points $E$ and $A$ are on the same side of line $\overrightarrow{BD}$. (justify)

(11) Points $E$ and $C$ are on the same side of line $\overrightarrow{BD}$. (justify)

(12) Points $A$ and $C$ are on the same side of line $\overrightarrow{BD}$. (justify)

(13) Point $C$ is in the interior of $\angle ABD$. (justify)

(14) $m(\angle ABC) + m(\angle CDB) = m(\angle ABD)$. (justify)

(15) $m(\angle ABC) = m(\angle ABD) - m(\angle CDB)$. (justify)

(16) $m(\angle ABC) < m(\angle ABD)$. (Here we have used the fact that $m(\angle CBD)$ is positive.)
The previous statement contradicts the fact that \( m(\angle ABD) = r = \frac{1}{2} m(\angle ABD) \). Therefore our assumption in step (8) was wrong. Points \( D \) and \( A \) cannot lie on opposite sides of line \( \overrightarrow{BC} \).

**Conclusion of cases**

(18) Since Cases (i) and (ii) both lead to contradictions, we conclude that only Case (iii) is possible. That is, points \( D \) and \( A \) lie on the same side of line \( \overrightarrow{BC} \).

**Part 3: Show that ray \( \overrightarrow{BD} \) is a bisector of \( \angle ABC \).**

(19) Point \( D \) is in the interior of \( \angle ABC \). (justify)

(20) \( m(\angle ABD) = m(\angle DBC) \). (justify. This will take 2 or 3 steps)

(21) Ray \( \overrightarrow{BD} \) is a bisector of \( \angle ABC \). (justify)

**Part 4: Show that ray \( \overrightarrow{BD} \) is the only bisector of \( \angle ABC \).**

(22) Suppose that ray \( \overrightarrow{BD'} \) is a bisector of \( \angle ABC \).

(23) Point \( D' \) is in the interior of \( \angle ABC \) and \( m(\angle ABD') = m(\angle D'BC) \). (justify)

(24) \( m(\angle ABD') = \frac{1}{2} m(\angle ABC) \). (justify)

(25) Points \( D' \) and \( C \) are on the same side of line \( \overrightarrow{AB} \). (justify)

(26) Point \( D' \) is in half-plane \( H_C \). (justify)

(27) Ray \( \overrightarrow{BD'} \) is the same ray as \( \overrightarrow{BD} \). (justify)

**End of Proof**
Drill for Section 6.5: The Angle Side Angle Congruence Theorem

Justify the steps in the proof of the following theorem. Draw a picture to illustrate. (Your justifications may refer to any prior theorem and to Axioms <N1> through <N10>.

Theorem 49: the ASA Congruence Theorem
In Neutral Geometry, if there is a correspondence between parts of two triangles such that two angles and the included side of one triangle are congruent to the corresponding parts of the other triangle, then all the remaining corresponding parts are congruent as well, so the correspondence is a congruence and the triangles are congruent.

Proof
(1) Suppose that $\triangle ABC$ and $\triangle DEF$ are given such that $\angle ABC \cong \angle DEF$ and $BC \cong EF$ and $\angle BCA \cong \angle EFD$.
(2) There exists a point $G$ on ray $ED$ such that $EG \cong BA$. (justify)

(3) $\triangle GEF \cong \triangle ABC$. (justify)

(4) $\angle EFG \cong \angle BCA$. (justify)

(5) $\angle EFG \cong \angle EFD$. (justify)

(6) Points $D$ and $G$ are on the same side of line $EF$. (justify)

(7) Ray $FD$ must be the same ray as $FG$. (justify)

(8) Line $EF$ can only intersect line $DE$ at a single point. (justify)

(9) Points $D, G$ must be the same point.
(10) $\triangle DEF \cong \triangle ABC$. (justify)

End of proof
Drill for Section 6.6: The Neutral Exterior Angle Theorem

Justify the steps in the proof of the following theorem:

Theorem 54: Neutral Exterior Angle Theorem
In Neutral Geometry, the measure of any exterior angle is greater than the measure of either of its remote interior angles.

Proof
(1) Suppose that a triangle and an exterior angle are given.

Part I: Show that the measure of the given exterior angle is larger than the measure of the remote interior angle that the exterior point does not lie on.
(2) Label the points so that the triangle is called \( \triangle ABC \) and the exterior angle is \( \angle CBD \).
   (make a drawing) Observe there are two remote interior angles, \( \angle BAC \) and \( \angle BCA \).
   Point \( D \) lies on side \( \overline{AB} \) of \( \angle BAC \). Point \( D \) does not lie on either of the sides of angle \( \angle BCA \).

(3) There exists a point \( E \) that is the midpoint of side \( \overline{BC} \). (justify) (update drawing)

(4) \( \overline{EB} \cong \overline{EC} \). (justify)

(5) There exists a point \( F \) such that \( A * E * F \). (justify) (update drawing)

(6) There exists a point \( G \) on ray \( \overrightarrow{EF} \) such that \( \overline{EG} \cong \overline{EA} \). (justify) (update drawing)

(7) \( \angle AEC \cong \angle GEB \). (justify)

(8) \( \triangle AEC \cong \triangle GEB \). (justify)
Make observations about angles
(9) $\angle ACE \cong \angle GBE$. (justify)

(10) Point $G$ is in the interior of $\angle DBC$. (justify)

(11) $m(\angle DBC) = m(\angle DBG) + m(\angle GBC)$. (justify)

(12) $m(\angle DBC) > m(\angle GBC)$. (justify)

(13) $m(\angle DBC) > m(\angle ACB)$. (justify)

Part II: Show that the measure of the given exterior angle is also larger than the measure of the remote interior angle that the exterior point does lie on.
(14) There exists a point $H$ such that $C \ast B \ast H$. (justify) (update drawing)

(15) Observe that $\angle ABH$ is an exterior angle for $\triangle ABC$ and also observe that the point $H$ does not lie on the remote interior angle $\angle BAC$.

(16) $m(\angle ABH) > m(\angle BAC)$ (by statements identical to statements (2) through (14), but with points $A, C, D$ replaced in all statements with points $C, A, H$.)
(17) $m(\angle ABH) = m(\angle CBD)$. (justify)

(18) $m(\angle CBD) > m(\angle BAC)$. (justify)

End of proof
Drill 1 for Section 7.2: The AAS Congruence Theorem

Justify the steps in the proof of the following theorem:

Theorem 65: the Angle-Angle-Side (AAS) Congruence Theorem for Neutral Geometry

In Neutral Geometry, if there is a correspondence between parts of two right triangles such that two angles and a non-included side of one triangle are congruent to the corresponding parts of the other triangle, then all the remaining corresponding parts are congruent as well, so the correspondence is a congruence and the triangles are congruent.

Proof

(1) Suppose that in Neutral Geometry, triangles $\triangle ABC$ and $\triangle DEF$ have $\angle A \cong \angle D$ and $\angle B \cong \angle E$ and $BC \cong EF$. (make a drawing)

(2) There exists a point $G$ on ray $BA$ such that $BG \cong ED$. (justify)

(3) $\triangle GBC \cong \triangle DEF$. (justify)

(4) $\angle CGB \cong \angle FDE$. (justify)

(5) $\angle CGB \cong \angle CAB$. (justify)

(6) There are three possibilities for where point $G$ can be on ray $BA$.

(i) $A \ast G \ast B$.

(ii) $G \ast A \ast B$.

(iii) $G = A$. 

Case (i) $A \ast G \ast B$.
(7) Suppose $A \ast G \ast B$. (make a drawing) Then $\angle CGB$ is an exterior angle for $\triangle CGA$, and $\angle CAB$ is one of its remote interior angles.

(8) $m(\angle CGB) > m(\angle CAB)$ (justify)

(9) We have reached a contradiction (explain the contradiction). Therefore, the assumption in step (6) was wrong.

Case (ii) $G \ast A \ast B$.
(10) – (12) Suppose $G \ast A \ast B$. (make a drawing, fill in the details to show that we reach a contradiction)

Conclusion of Cases
(13) Since Cases (i) and (ii) lead to contradictions, we conclude that only Case (iii) is possible. That is, it must be that $G = A$. Therefore, $\triangle ABC \cong \triangle DEF$.

End of proof
Drill 2 for Section 7.2: The Hypotenuse Leg Congruence Theorem

Justify the steps in the proof of the following theorem:

Theorem 66: the Hypotenuse Leg Congruence Theorem for Neutral Geometry
In Neutral Geometry, if there is a correspondence between parts of two right triangles such that the hypotenuse and a side of one triangle are congruent to the corresponding parts of the other triangle, then all the remaining corresponding parts are congruent as well, so the correspondence is a congruence and the triangles are congruent.

Proof
(1) Suppose that in Neutral Geometry, right triangles $\triangle ABC$ and $\triangle DEF$ have right angles at $\angle A$ and $\angle D$, and congruent hypotenuses $BC \cong EF$, and a congruent leg $AB \cong DE$, (Make a drawing.)

(2) There exists a point $G$ such that $C \ast A \ast G$ and $AG \cong DF$. (Justify. It will take three statements.) (Update your drawing.)

(3) $\triangle ABG \cong \triangle DEF$. (justify)

(4) $BG \cong EF$. (justify)

(5) $\angle AGB \cong \angle DFE$. (justify)

(6) $\angle AGB \cong \angle ACB$. (justify)

(7) $\angle ACB \cong \angle DFE$. (justify)

(8) $\triangle ABD \cong \triangle DEF$. (by 0, the AAS Congruence Theorem for Neutral Geometry)

End of proof