List of Axioms and Theorems from Appendix A

**Axiom A.1** Lines, planes, and space are sets of points. Space contains all points.

**Axiom A.2** Any two distinct points are on exactly one line. Every line contains at least 2 points.

**Axiom A.3** Any three noncollinear points are on exactly one plane. Each plane contains at least three noncollinear points.

**Axiom A.4** If two points of a line are in a plane, then the entire line is in the plane.

**Axiom A.5** In space, if two planes have a point in common, then the planes have an entire line in common.

**Axiom A.6** In space, there exist at least four points that are noncoplanar.

**Axiom A.7** (Ruler) The points on a line can be put in one-to-one correspondence with the real numbers

**Axiom A.8** (Plane Separation) Each line in a plane separates all the points of the plane that are not on the line into two nonempty sets, called the half planes, with the following properties:
1. The half planes are disjoint convex sets.
2. If P is in one half plane and Q is in the other half plane, then segment PQ intersects the line that separates the plane.

**Axiom A.9** (Angle Measurement) Every angle has a measure that is a real number between 0 and 180 (inclusive). The number 180 corresponds to a straight angle, and the number corresponds to an angle where the two rays of the angle are the same ray.

**Axiom A.10** (Angle Construction) Let ray AB be on the edge of a half plane. For every real number r such that 0 < r < 180, there exists exactly one ray AC, with point C in the half plane, such that the measure of angle CAB is r.

**Axiom A.11** (Angle Addition) If D is a point in the interior of angle ABC, then m(BAD)+m(DAC)=m(BAC)
Theorem A.1 In Neutral Geometry, if two distinct lines intersect, they intersect in exactly one point.

Theorem A.2 (Pasch) In Neutral Geometry, if a line intersects a side of a triangle and does not intersect any of the vertices, then it also intersects another side of the triangle.

Theorem A.3 (The Crossbar Theorem) In Neutral Geometry, the following are equivalent
   1. Ray CD is between rays CA and CB
   2. Ray CD intersects side AB of triangle ABC at a point between A and B

List of Axioms and Theorems from Chapter 1

Axiom (The Euclidean Parallel Postulate) (Presented in intro to Chapter 1) For any line $L$ and any point $P$ not on $L$, there is not more than one line $M$ that passes through $P$ and is parallel to $L$.

Axiom 1.1 (Presented in Section 1.2) SAS congruence condition

Axiom (The Parallel Postulate) (Playfair’s Axiom) (Presented in Section 1.3) For any line $L$ and any point $P$ not on $L$, there is exactly one line $M$ that passes through $P$ and is parallel to $L$.

Theorem 1.1 In Neutral Geometry
   1. Supplements of congruent angles are congruent
   2. Complements of congruent angles are congruent
   3. The sum of the measures of two angles in a linear pair is 180

Theorem 1.2 In Neutral Geometry, vertical angles are congruent

Theorem 1.3 (The isosceles triangle theorem) In Neutral Geometry, $CS \cong CA$

Theorem 1.4 In Neutral Geometry, The ASA congruence condition holds

Unnumbered theorem: In Neutral Geometry, $CA \cong CS$

Theorem 1.5 In Neutral Geometry, the median to the base of an isosceles triangle is the perpendicular bisector as well as the angle bisector of the angle opposite the base.
**Theorem 1.6** In Neutral Geometry, every point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment.

**Corollary 1.2** In Neutral Geometry, a point is equidistant from the endpoints of a segment iff it is on the perpendicular bisector of the segment.

**Corollary 1.3** In Neutral Geometry, if each of two points is equidistant from the endpoints of a segment, then the line through these points is the perpendicular bisector of the segment.

**Theorem 1.7** In Neutral Geometry, the main diagonal of a kite (the diagonal that connects the vertices where the congruent sides intersect) bisects the angles at these vertices and is the perpendicular bisector of the other diagonal.

**Corollary 1.4** In Neutral Geometry, the diagonals of a rhombus are perpendicular bisectors of each other, and each bisects a pair of opposite angles.

**Theorem 1.8** In Neutral Geometry, the SSS congruence condition.

**Theorem 1.9** (Hypotenuse-Leg Congruence Condition) In Neutral Geometry, if the hypotenuse and leg of one right triangle are congruent to the hypotenuse and leg of another right triangle, then the triangles are congruent.

**Theorem 1.10** (Exterior Angle Theorem) In Neutral Geometry, an exterior angle of a triangle is greater than either of the remote interior angles.

**Corollary 1.5** In Neutral Geometry, through a point not on a line, there is a unique perpendicular to the line.

**Theorem 1.11** In Neutral Geometry, the Hypotenuse-Acute Angle congruence condition holds.

**Theorem 1.12** In Neutral Geometry, a point is on the angle bisector of an angle iff it is equidistant from the sides of the angle.

**Theorem 1.13** In Neutral Geometry, given two non-congruent sides in a triangle, the angle opposite the longer side is greater than the angle opposite the shorter side (BS $\Rightarrow$ BA)
**Theorem 1.14** In Neutral Geometry, given two non-congruent angles in a triangle, the side opposite the larger angle is longer than the side opposite the smaller angle (BA ≥ BS)

**Theorem 1.15** (Triangle Inequality) In Neutral Geometry, the sum of the lengths of any two sides of a triangle is greater than the length of the third side.

**Theorem 1.16** In Neutral Geometry, if two lines in the same plane are each perpendicular to a third line in that plane, then they are parallel.

**Theorem 1.17** In Neutral Geometry, if two lines are cut by a transversal and a pair of corresponding angles is congruent (or a pair of alternate interior angles is congruent), then the lines are parallel.

**Big Big Theorem** that is not presented in the book. For any line $L$ and any point $P$ not on $L$, there exists at least one line $M$ that passes through $P$ and is parallel to $L$.

**Theorem 1.18** In Euclidean Geometry, if two parallel lines are cut by a transversal, then a pair of corresponding angles is congruent.

**Theorem 1.19** In Euclidean Geometry, two lines in a plane are parallel if and only if a pair of corresponding angles formed by a transversal is congruent.

**Theorem 1.20** In Euclidean Geometry, two lines in a plane are parallel if and only if a pair of alternate interior angles formed by a transversal is congruent.

**Theorem 1.21** In Euclidean Geometry, two lines in a plane are parallel if and only if a pair of interior angles on the same side of a transversal is supplementary.

**Theorem 1.22** In Euclidean Geometry, the sum of the measures of the interior angles of a triangle is 180°.

**Theorem 1.23** In Euclidean Geometry, every Saccheri quadrilateral is a rectangle.

**Theorem 1.24** In Euclidean Geometry, the measure of an exterior angle is equal to the sum of the measures of its two remote angles.

**Theorem 1.25** In Euclidean Geometry, if a transversal is perpendicular to one of two parallel lines, then it is also perpendicular to the other.
Theorem 1.26 In Euclidean Geometry, in a parallelogram:
1. Each diagonal divides the parallelogram into two congruent triangles.
2. Each pair of opposite sides is congruent.
3. The diagonals bisect each other.

Theorem 1.27 In Euclidean Geometry,
1. A quadrilateral in which each pair of opposite sides is congruent is a parallelogram.
2. A quadrilateral in which the diagonals bisect each other is a parallelogram.
3. A quadrilateral in which each pair of opposite angles is congruent is a parallelogram.
4. A quadrilateral in which a pair of opposite sides is parallel and congruent is a parallelogram.

Theorem 1.28 In Euclidean Geometry, parallel projections preserve betweenness.

Theorem 1.29 In Euclidean Geometry, a parallel projection preserves congruence of segments belonging to the same line.

Corollary 1.6 In Euclidean Geometry, if three or more parallel lines intercept congruent segments on one transversal, then they intercept congruent segments on any other transversal.

Theorem 1.30 In Euclidean Geometry, a line through the midpoint of one side of a triangle and parallel to the second side bisects the third side.

Theorem 1.31 (The Midsegment Theorem) In Euclidean Geometry, the segment connecting the midpoints of two sides of a triangle is parallel to the third side and half as long as that side.

Theorem 1.31 (Property of Medians) In Euclidean Geometry, the three medians of a triangle are concurrent in a point whose distance to a vertex is two-thirds of the length of the median from that vertex.

List of Propositions and Theorems from Chapter 2

Remark about the four propositions at the beginning of Chapter 2: The book presents four “Propositions” on pages 76-77. These are statements that the book says are easily proved, and so the proofs are omitted. That means that we are to treat these statements as axioms.
Remark about Proposition 2.1: It is dumb to even state this proposition, because it is just a restatement of the definition of congruent arcs, found two sentences earlier on the same page.

Proposition 2.1 In Euclidean Geometry, two minor arcs in the same circle or in congruent circles are congruent if and only if their central angles are congruent.

Remark about Propositions 2.2, 2.3, and 2.4: The proof of these propositions are not trivial. It would have been better for the book to present these statements as theorems and give proofs.

Proposition 2.2 In Euclidean Geometry, in a circle or in congruent circles, two arcs are congruent if and only if the corresponding chords are congruent.

Proposition 2.3 In Euclidean Geometry, a diameter perpendicular to a chord bisects the chord and bisects each of the arcs determined by the chord.

Proposition 2.4 In Euclidean Geometry, if two circles intersect in exactly two points, then the line through their centers is the perpendicular bisector of their common chord. 
Remark: The boldface words “in exactly two points” are essential. The book omits them.

Theorem 2.1 In Euclidean Geometry, the measure of the angle formed by a tangent and a chord equals half the measure of the intercepted arc.

Theorem 2.2 (The Inscribed Angle Theorem) In Euclidean Geometry, the measure of an inscribed angle equals half the measure of the intercepted arc.

Corollary 2.1 In Euclidean Geometry, in any circle, all the inscribed angles intercepting the same arc are congruent.

Remark about Theorem 2.3: The book butchers the wording of this theorem, and presents a picture that has errors. Here is a rewording of the theorem.

Theorem 2.3 In Euclidean Geometry, if A,B,C are non-collinear points and P is a point that is on the same side of line(BC) as point A, then the following are equivalent: 
(1) Angle(BPC) is congruent to angle(BAC).
(2) Point P is on arc(BAC).
**Corollary 2.2** In Euclidean Geometry, any angle inscribed in a semicircle is a right angle.

**Theorem 2.4** In Euclidean Geometry, a necessary and sufficient condition for a quadrilateral to be cyclic is that the sum of the measures of a pair of opposite angles is $180^\circ$.

**Theorem 2.5** In Euclidean Geometry, the angle bisectors of any triangle intersect at a single point $O$ that is equidistant from the three sides. That means that a circle can be drawn that is centered at point $O$ and that touches each of the three sides in exactly one point. In other words, point $O$ is the center of the inscribed circle, and the radius of the inscribed circle is equal to the distance from point $O$ to any of the three sides.

**Theorem 2.6** In Euclidean Geometry, from a point in the exterior of a circle, the two tangent segments are congruent.

**Theorem 2.7** In Euclidean Geometry, a circle can be inscribed in a quadrilateral if and only if the angle bisectors of the four angles of the quadrilateral are concurrent.

**Remark about Theorem 2.8:** The book uses the word “circumscribable” in the following way: They say that a polygon is circumscribable if a circle can be inscribed inside the polygon. This is a rather bizarre way to define the word, one that goes against more common usage. I will use alternate wording

**Theorem 2.8** In Euclidean Geometry, if a circle can be inscribed in a quadrilateral then the sum of the lengths of two opposite sides of the quadrilateral equals the sum of the lengths of the two remaining opposite sides.

**Theorem 2.9** (Converse of Theorem 2.8) In Euclidean Geometry, if the sum of the lengths of two opposite sides of a quadrilateral equals the sum of the lengths of the two remaining opposite sides, then a circle can be inscribed in the quadrilateral.

**Theorem 2.10** (Combination of Theorems 2.8 and 2.9) In Euclidean Geometry, A circle can be inscribed in a quadrilateral if and only if the sum of the lengths of two opposite sides of the quadrilateral equals the sum of the lengths of the two remaining opposite sides.
Axiom 3.1 The area of any polygonal region (a polygon and its interior) is a unique positive real number.

Axiom 3.2 The area of any point or line segment is 0.

Axiom 3.3 Congruent polygonal regions have the same area.

Axiom 3.4 Area is additive. That is, if we can divide a figure into non-overlapping parts, or parts that share only lines or line segments or points, then its area is the sum of the areas of the parts.

Axiom 3.5 (Area of a square) A square of length $a$ units has area of $a^2$ square units.

Theorem 3.1 (Area of a Rectangle) In Euclidean Geometry, the area of a rectangle with sides of length $a$ and $b$ is $ab$.

Theorem 3.2 (Area of a Parallelogram) In Euclidean Geometry, the area of a parallelogram is the product of the length of a base and the corresponding height.

Theorem 3.3 (Area of a Triangle) In Euclidean Geometry, the area of a triangle is half the product of the length of a base and the corresponding height.

Theorem 3.4 (Area of a Trapezoid) In Euclidean Geometry, the area of a trapezoid whose bases have length $a$ and $b$ and whose height is $h$ is given by the formula $\text{area} = (1/2)(a + b)h$.

Theorem 3.5 (The Pythagorean Theorem) In Euclidean Geometry, in a right triangle with legs of length $a$ and $b$ and hypotenuse of length $c$, values of $a$, $b$, $c$ satisfy the equation $a^2 + b^2 = c^2$.

Theorem 3.6 (The Distance Formula) In Euclidean Geometry, the distance between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Theorem 3.6 In Euclidean Geometry, if two lines (neither of which is vertical) are perpendicular, then the products of their slopes is -1.
List of Axioms and Theorems from Chapter 4

**Theorem 4.1** Parallel projection preserves lengths of segments.

**Theorem 4.2** (The Side-Splitting Theorem)
If a line is parallel to a side of a triangle and intersects the other two sides in distinct points,
then the line splits these sides into proportional segments (the ratio of the segments on one side equals the ratio of the corresponding segments on the other side).

**Theorem 4.3** (The Converse of the Statement of the Side-Splitting Theorem)
If a line divides two sides of a triangle proportionally (the ratio of the segments on one side equals the ratio of the corresponding segments on the other side),
then the line is parallel to the third side.

**Theorem 4.4** (AA Similarity Condition for Triangles)
If two angles of one triangle are congruent to two angles of another triangle,
then the triangles are similar.

**Theorem 4.5** (SSS Similarity Condition for Triangles)
If the corresponding sides of two triangles are proportional,
then the triangles are similar.

**Theorem 4.6** (SAS Similarity Condition for Triangles)
If two triangles have two pairs of corresponding sides that are proportional and the included angles are congruent,
then the triangles are similar.

**Theorem 4.7**
If an interior angle of a triangle is bisected,
then the bisector divides the opposite side into segments whose lengths have the same ratio as the lengths of the other sides of the triangle.

**Theorem 4.8**
If \( \triangle ABC \) has exterior angle \( \angle BAE \) and a ray \( (A,D) \) bisects \( \angle BAE \) and intersects line \( (B,C) \) at point \( D \),
then \( DC/DB = AC/AB \).
Theorem 4.9 (The Circle of Apollonius)
The locus of all points $P$ for which the ratio of the distances from two fixed points $A$ and $B$ is constant is the circle of diameter $DD_1$, where points $D$ and $D_1$ divide segment $(A,B)$ harmonically into the ratio of that constant.

Theorem 4.10 In a right triangle, the altitude to the hypotenuse is the geometric mean of the segments into which it divides the hypotenuse.

Theorem 4.11
If two chords intersect in the interior of a circle, then the product of the lengths of the segments of one chord equals the product of the lengths of the segments of the other chord, and the common product equals $r^2 - d^2$, where $r$ is the radius of the circle and $d$ is the distance from the point of intersection of the chords to the center of the circle.

Theorem 4.12
If two secant lines containing chords $AB$ and $A_1B_1$ in a circle intersect at a point $P$ in the exterior of the circle, then $PA \cdot PB = PA_1 \cdot PB_1$.

Theorem 4.12
If a secant line containing a chord $AB$ in a circle and a line to that circle at point $C$ intersect at a point $P$ outside the circle, then $PA \cdot PB = PC^2$.

Corollary 4.1
If the line containing a chord $AB$ and a line tangent to a circle at point $T$ intersect at a point $P$ outside a circle, then $PA \cdot PB = PT^2$.

Theorem 4.13 The ratio of areas of similar triangles equals the square of the ratio of corresponding sides.

Theorem 4.14 The ratio of areas of similar polygons equals the square of the ratio of corresponding sides.

Theorem 4.15
If three similar polygons are constructed on the three sides of a right triangle, then the area of the polygon on the hypotenuse equals the sum of the areas of the polygons on the other sides.
Theorems from Chapter 6

**Theorem 6.1** The First Fundamental Theorem of Isometries
If $A, B, C$ are noncollinear points, and $A', B', C'$ are points such that $A'B' = AB$ and $B'C' = BC$ and $C'A' = CA$, then there exists a unique isometry that maps $A \rightarrow A'$ and $B \rightarrow B'$ and $C \rightarrow C'$. Furthermore, this isometry can be expressed as a composition of three or fewer reflections.

**Corollary 6.1** If two isometries have the same values (images) at three noncollinear points, then the isometries are in fact the same isometry.

**Theorem 6.2** The Second Fundamental Theorem of Isometries. Every isometry is equal to a composition of three or fewer reflections.

**Theorem 6.3** If lines $K$ and $L$ intersect at point $P$, then the composition $M_L \circ M_K$ is equivalent to the rotation $R_{P,2\theta}$ where $\theta$ is the directed angle from line $K$ to line $L$.

**Corollary 6.2** Any rotation $R_{P,\alpha}$ can be expressed in infinitely many ways as the composition of two reflections. The lines of reflection can be any pair of lines through $P$ such that the directed angle between the lines is $\alpha/2$.

**Theorem 6.4** If $K$ and $L$ are parallel lines, then the composition $M_L \circ M_K$ is equivalent to the translation $\tau_v$ where $v$ is a vector from $K$ to $L$ that is perpendicular to both lines.

**Corollary 6.3** A translation $\tau_w$ can be expressed in infinitely many ways as a composition of two reflections in parallel lines such that $w/2$ is the vector from one line to the other and is perpendicular to both lines.

**Theorem 6.5** If a vector $v$ is parallel to a line $M$, then $M_M \circ \tau_v = \tau_v \circ M_M$.

**Theorem 6.6** A composition of reflections in three nonconcurrent lines, at least two of which intersect, is a glide reflection.

**Theorem 6.7** If $R_{A,\alpha}$ and $R_{B,\beta}$ are rotations with centers at distinct points $A$, $B$, and $\alpha + \beta$ is not a multiple of $360^\circ$. then the composition $R_{B,\beta} \circ R_{A,\alpha}$ is a rotation by the angle $\alpha + \beta$. If $\alpha + \beta$ is a multiple of $360^\circ$, then the composition is a translation.

**Theorem 6.8** (Napoleon’s Theorem) Don’t try to take over the world. You’ll end up imprisoned on an island for the rest of your life.