Math 330A Homework Set #3, Due Tuesday, January 26, 2010

[1] (Variation on Exercise 1.2#6) Justify steps in the proof of this theorem:

Theorem: In Euclidean Geometry, in any triangle, the perpendicular bisectors of the three sides are concurrent.

Proof

(1) Suppose that \( \triangle ABC \) is a triangle in Euclidean Geometry.

Show that the perpendicular bisector of side \( AB \) exists.
(2) The midpoint of side \( AB \) exists. (justify: ____________________________________________________________________)

Label the midpoint \( M \).
(3) There exists a line \( K \) that passes through point \( M \) and is perpendicular to side \( AB \). (justify: ____________________________________________________________________).

Observe that line \( K \) is the perpendicular bisector of side \( AB \).

Show that the perpendicular bisector of side \( BC \) exists.
(4) Steps just like steps 2 and 3 could be used to show that side \( BC \) has a midpoint \( N \) and that there exists a line \( L \) that is the perpendicular bisector of side \( BC \).

Show that lines \( K \) and \( L \) meet
(5) The fact that lines \( K \) and \( L \) meet can be proved, but it requires the Euclidean Parallel Postulate. Just take it as a given that lines \( K \) and \( L \) meet at a point \( O \).

Discuss the distances from point \( O \) to the three vertices.
(6) We know that \( AO = BO \) (justify: ____________________________________________________________________)

(7) We know that \( BO = CO \) (justify: ____________________________________________________________________)
(8) Therefore, \( AO = CO \) (justify: ____________________________________________________________________)

Show that point \( O \) is actually on the perpendicular bisector of side \( CA \), as well.
(9) Because \( AO = CO \), we know that point \( O \) must be on the perpendicular bisector of side \( CA \). (justify: ____________________________________________________________________)

In conclusion, we have shown that there exists a point \( O \) that all three perpendicular bisectors pass through.
End of Proof
[2] (Variation on Exercise 1.2#6) Given the acute triangle below, construct the circle that circumscribes it.

Hint: Use what you learned in exercise [1]. That is, start by constructing the perpendicular bisectors of two of the sides of the triangle. You know that they will meet at some point that you can call \( O \), and you know from problem [1] that this point \( O \) would also be on the perpendicular bisector of the third side if it were drawn. (You don’t need to draw that third perpendicular bisector.) Therefore, this point \( O \) must be equidistant from the three vertices of the triangle. Construct the circle that is centered at \( O \) and passes through \( A \).

Remember that you will need to do two things:

- Produce a list of instructions for the construction consistent with the rules.
- Do the construction on paper with your straight-edge and non-collapsing compass. (You can draw right on this page. List the instructions on a separate sheet.)
[3] (Variation on Exercise 1.2#6) Justify the steps in the proof of this theorem:

**Theorem:** In Neutral Geometry, in any triangle, the angle bisectors of the three angles are concurrent.

Proof

(1) Suppose that ΔABC is a triangle in Neutral Geometry. Show that the angle bisector of angle A exists.

(2) We can measure the size of angle A. The result is a number x such that 0 < x < 180. (justify ________________________________________)

(3) There exists a Ray(AD) in the interior of ∠A such that the m(∠BAD) = x/2. (justify ________________________________________)

Observe that Ray(AD) is the bisector of ∠A.

Show that the angle bisector of angle B exists.

(4) Steps just like steps 2 and 3 could be used to show that there exists a ray BE that is the bisector of angle B.

Show that rays AD and BE meet

(5) Ray AD intersects segment BC at a point that we can label F. (justify: ________________________________________)

(6) Ray BE intersects segment AF at a point that we can label G. (justify: ________________________________________)

Observe that point G is on ray AD and also on ray BE.

Discuss the distances from point G to the three sides of the triangle.

(7) We know that point G is equidistant from lines CA and AB (justify: ________________________________________)

(8) We know that point G is equidistant from lines AB and BC (justify: ________________________________________)

(9) Therefore point G is equidistant from lines BC and CA. (justify: ________________________________________)

Show that point G is actually on the angle bisector of ∠BCA, as well

(10) Because point G is equidistant from lines BC and CA we know from theorem _____________ that point G must be on the angle bisector of ∠BCA.

In conclusion, we have shown that there exists a point G that all three angle bisectors pass through.

End of Proof
[4] (Variation on Exercise 1.2#6) Given the acute triangle below, construct a circle that is inscribed in it. (Remember that you have to make a list of instructions.) Hint: Use what you learned in exercise [3]. That is, start by constructing the bisectors of two of the angles of the triangle. From problem [2], you know that they will meet at some point that you can call $G$, and you know that this point $G$ would also be on the bisector of the third angle if it were drawn. (You don’t need to draw it.) Therefore, this point $G$ must be equidistant from the three sides of the triangle. The circle that you need to create must be centered at $G$. But in order to draw the circle, you will need to know its radius. For that, construct a line that passes through $O$ and is perpendicular to one of the sides of the triangle. Construct a point $H$ at the intersection of the perpendicular and the side of the triangle. Use $OH$ as the radius of your circle.
[5] (variation on exercise 1.2#10a) Justify steps in the proof of Theorem 1.7.

*Theorem 1.7* In Neutral Geometry, in any convex kite, the angle bisectors are concurrent

Proof

(1) Let $ABCD$ be a convex kite with $BA = BC$ and $DA = DC$

Show that the main diagonal bisects angles $B$ and $D$

(2) $\Delta BAD \cong \Delta BCD$ (justify: ____________________________________)  

(3) $\angle ABD \cong \angle CBD$ (justify: ____________________________________)

This tells us that the main diagonal bisects $\angle B$.

(4) $\angle ADB \cong \angle CDB$ (justify: ____________________________________)  

This tells us that the main diagonal bisects $\angle D$.

Introduce the bisector of angle $A$ and the point $O$

(5) A ray $AE$ exists that bisects $\angle A$. (justify: ______________________________)

(6) Ray $AE$ intersects segment $BD$ (justify: ______________________________)

Let $O$ be the point of intersection of Ray $AE$ and segment $BD$. Note that so far, we know that the bisectors of angles $B$, $D$, and $A$ intersect at point $O$.

Introduce some distances

(7) Let $d_1$ be the distance from point $O$ to line $AD$,  
let $d_2$ be the distance from point $O$ to line $AB$,  
let $d_3$ be the distance from point $O$ to line $BC$  
let $d_4$ be the distance from point $O$ to line $CD$.

Discuss known equalities among some of these distances

(8) $d_3 = d_2$ (justify: ______________________________)

(9) $d_2 = d_1$ (justify: ______________________________)

(10) $d_1 = d_4$ (justify: ______________________________)

(11) $d_3 = d_4$ (justify: ______________________________)

Wrap-up

(12) Point $O$ must be on the bisector of angle $C$. (justify: ______________________________)

We conclude that the bisectors of all four angles intersect at point $O$.

End of Proof
[6] (Variation on Exercise 1.2#10b) Create a geogebra drawing to illustrate the fact that given any convex kite (ABCD), a circle can be inscribed in the kite.

- In your drawing, points A, B, and D should be free. Point C and all other parts of the drawing should be dependent.
- Somewhere in the upper right of the drawing, insert a textbox with the text H3.Lastname using your last name.
- Save the drawing. Give it the filename H3.Lastname
- Send me the drawing in an e-mail
  - Recipients:
    - me: Mark.Barsamian.1@ohio.edu
    - you: your OU e-mail address
  - Subject line: Geometry H3 Lastname
  - Attachment: Your file called H3.Lastname
  - Body of the message: Geometry Homework 3 from Your Name.

Hint: Use the theorem that you proved in [5]. That is, start by constructing the bisectors of two of the angles of the kite. From problem [5], you know that they will meet at some point that you can call $O$, and you know that this point $O$ would also be on the bisector of the third and fourth angles if they were drawn. (You don’t need to draw them.) Therefore, this point $O$ must be equidistant from the four sides of the kite. The circle that you need to create must be centered at $O$. But in order to draw the circle, you will need to know its radius. For that, construct a line that passes through $O$ and is perpendicular to one of the sides of the kite. Construct a point $H$ at the intersection of the perpendicular and the side of the kite. Use $OH$ as the radius of your circle.
[7] (Variation on Exercise 1.2#14a) Given the scalene triangle \( \triangle ABC \) below, construct another triangle \( \triangle A_1B_1C_1 \) congruent to it such that your constructed point \( B_1 \) lies on the given ray \( A_1D \).

Remember that you will need to do two things:
- Produce a list of instructions for the construction consistent with the rules.
- Do the construction on paper with your straight-edge and non-collapsing compass. (You can draw right on this page. List the instructions on a separate sheet.)
[8] Find the sum of the measures of the interior angles of a convex $n$-gon using two methods:
   (a) Use the picture and suggestion in textbook Exercise 1.3#2b.
   (b) Use the picture and suggestion in textbook Exercise 1.3#3.

[9] The goal is to solve textbook problem 1.3#6abd. (Skip part c) I will give you some additional hints.
For part (a) of problem #6 in the book, find an equation that gives $x$ in terms of $y$ and $z$. (Hint: Through the point that is the vertex of $\angle y$, draw a line that is parallel to line $a$. Look at the way this line breaks up $y$ into two smaller angles. Think about two smaller angles: What do they add up to? What angles are they each congruent to? Use this information to get an equation that gives $x$ in terms of $y$ and $z$.)

For part (b) of problem #6 in the book, find an equation that gives $\alpha_4$ in terms of $\alpha_3$, $\alpha_2$, and $\alpha_1$. (Hint: Through the point that is the vertex of angle $\alpha_3$, draw a line that is parallel to line $a$. Look at the way this line breaks up angle $\alpha_3$ into two smaller angles. The upper small angle is congruent to angle $\alpha_4$. The lower small angle has a measure that you can compute using the formula from part (a). Using this information, and angle addition, you can get an equation that gives $\alpha_4$ in terms of $\alpha_3$, $\alpha_2$, and $\alpha_1$.)

For part (d) of the problem in the book, notice the pattern from parts (a) and (b).

[10] This is a clarification of problem 1.3#9 in the book. Find a formula that expresses the measure of $\angle D$ in terms of the measure of $\angle A$. 