Class Drill 1.2.01 (done in class on Monday, January 11, 2010)

Shown below is the book’s “Proof 1” of the Isosceles Triangle Theorem. Justify the steps.

Theorem 1.3 The Isosceles Triangle Theorem (CS ➞ CA): If two sides of a triangle are congruent, then the angles opposite those sides are congruent.

Proof:

1. Let Δ(ABC) be a triangle such that AB = AC.
2. Some ray(AE) exists that bisects angle(BAC). justify:________________
   ______________________________________________________

3. Ray(AE) intersects segment(BC) at a point that we can call D. (justify) ________________________________

4. Segment(AD) is congruent to itself. (justify) ________________________________

5. Δ(BAD) is congruent to Δ(CAD). (justify) ________________________________

6. angle(ABD) is congruent to angle(CAD). (justify) ________________________________

End of Proof
Class Drill 1.2.02 (done in class on Tuesday, January 12, 2010)

Shown below is a proof of the ASA condition. It is different from the book’s proof.

**Theorem 1.4 The Angle-Side-Angle (ASA) Condition:** If two angles and the included side of one triangle are congruent to two two angles and the included side of another triangle, then the triangles are congruent.

**Proof:**

1. Let ∆(ABC) and ∆(A_1B_1C_1) be triangles such that ∠A ≅ ∠A_1, side AB ≅ side A_1B_1, and ∠B ≅ ∠B_1.
   2. There is a point C_2 on ray(A_1C_1) such that A_1C_2 = AC. (justify):
   
   
   3. ∆(CAB) ≅ ∆(C_2A_1B_1). (justify)
   
   
   4. ∠C_2B_1A_1 ≅ ∠CBA. (justify)
   
   
   5. ∠C_2B_1A_1 ≅ ∠C_1B_1A_1. (justify)
   
   
   6. Ray(B_1C_1) must be the same ray as ray(B_1C_2). (justify)
   
   
   7. Point C_2 must be the same point as point C_1. (justify)
   
   
   8. ∆(ABC) ≅ ∆(A_1B_1C_1) (justify)
   
   
End of Proof
Justify the proof of the following Theorem.

Theorem 1.13: Given two non-congruent sides in a triangle, the angle opposite the longer side is bigger than the angle opposite the shorter side. ($BS \Rightarrow BA$).

Proof:

1. Let $\triangle(ABC)$ and be a triangle such that $AB > AC$.
2. There is a point D on ray(AB) such that $AD = AC$.
   (justify):_________________________________________ _____
   _____________________________________________________ _____
3. $m(\angle ACB) > m(\angle ACD)$. (justify) __________________________
   _____________________________________________________ _____
4. $m(\angle ACD) = m(\angle ADC)$. (justify) __________________________
   _____________________________________________________ _____
5. $m(\angle ADC) > m(\angle ABC)$. (justify) __________________________
   _____________________________________________________ _____
6. $m(\angle ACB) > m(\angle ABC)$. (justify)_____________________________
   _____________________________________________________ _____

End of Proof
Justify the proof of the following Theorem.

*Theorem:*

*Given two non-congruent angles in a triangle, the side opposite the larger angle is longer than the side opposite the smaller angle (BA \( \implies \) BS).*

*Remark: The statement of this theorem is the converse of the statement of Theorem 1.13.*

**Proof:**

1. Let \( \Delta(ABC) \) and be a triangle such that \( m(\angle ACB) > m(\angle ABC) \).
2. The statement \( m(\angle ABC) > m(\angle ACB) \) is false. (justify): __________

3. The statement \( AC > AB \) is false. (justify) __________________________

4. The statement \( m(\angle ABC) = m(\angle ACB) \) is false. (justify): ________

5. The statement \( AC = AB \) is false. (justify) __________________________

6. The statement \( AB > AC \) is true. (justify) __________________________

End of Proof
Here is a re-statement of the book’s Theorem 1.5 about medians & altitudes to the base of isosceles triangle:

Theorem 1.5 Given Neutral Geometry, $\triangle ABC$ with $\text{side}(AB) \cong \text{side}(AC)$, and $D$ a point on line($BC$)

Claim: The Following Are Equivalent (TFAE)

(1) Point $D$ is the midpoint of segment($BC$). (So $AD$ is the median to the base $BC$)

(2) $\angle DAB \cong \angle DAC$. (So ray($AD$) is the bisector of angle $A$)

(3) $\angle ADB$ is a right angle. (So $AD$ is the altitude to the base $BC$)

Fill in the proof in the outline provided below. (Part 1 is below; Parts 2 and 3 are on back) In Part 3, you will justify steps.

Proof Part 1: Proof that (1) $\implies$ (2).
Proof Part 2: Prove that (2) $\Rightarrow$ (3).

Proof Part 3: Prove that (3) $\Rightarrow$ 1.

1. Let $\triangle(ABC)$ and be a triangle such that side (AB) $\cong$ side(AC), and D is a point on line(BC) such that $\angle ADB$ is a right angle. (So segment(AD) is the altitude to the base BC.)
2. There is a point M that is the midpoint of segment(BC). (justify): __________________________
3. Ray(AM) is the bisector of angle A. (justify) __________________________
4. Segment(AM) is the altitude to the base BC. (justify): ____________
5. Line(AM) must be the same line as line(AD). (justify) ____________
6. Point D must be the same point as the midpoint M (justify)________

End of Proof
Theorem 1.8 Side, Side, Side (SSS) Congruence Condition.

In Neutral Geometry, if there is a correspondence between the vertices of two triangles such that the three sides of one triangle are congruent to the corresponding sides of the other triangle, then the triangles are congruent.

Proof:

1. Let $\triangle(ABC)$ and $\triangle(A_1B_1C_1)$ be triangles such that $AB = A_1B_1$, $BC = B_1C_1$, and $CA = C_1A_1$.

Create a copy of $\triangle(A_1B_1C_1)$ on one side of $\triangle(ABC)$.

2. There is a real number $r$ that is the measure of $\angle B_1A_1C_1$. (justify): __________________________________________________________________________

3. Line AC creates two half planes. In the half-plane that does not contain point B, there is a point X such that ray(AX) creates $\angle CAX$ such that $m(\angle CAX) = r$. (justify) __________________________________________________________________________

4. There is a point $B_2$ on ray(AX) such that $AB_2 = A_1B_1$.

Show that the new triangle is congruent to one of the old triangles.

5. $\triangle(AB_2C) \cong \triangle(A_1B_1C_1)$. (justify): __________________________________________________________________________

Show that the new triangle is congruent to the other old triangle.

6. Quadrilateral (ABCB2) is a kite. (justify): __________________________________________________________________________

7. line(AC) bisects $\angle BAB_2$. (justify) __________________________________________________________________________

8. $\triangle(ABC) \cong \triangle(AB_2C)$. (justify): __________________________________________________________________________

Conclusion

9. $\triangle(ABC) \cong \triangle(A_1B_1C_1)$. (justify): __________________________________________________________________________

End of Proof
Theorem 1.9 (The Hypotenuse-Leg Congruence Condition)
In Neutral Geometry, if the hypotenuse and one leg of one right triangle are congruent to the hypotenuse and one leg of another right triangle, then the triangles are congruent.

Justify the steps in the following proof. (Draw pictures where indicated.)

Proof
1. Let \( \triangle ABC \) and \( \triangle DEF \) be triangles such that angles \( \angle A \) and \( \angle D \) are right angles, \( BC = EF \), and \( BA = ED \). (Draw a picture.)

Prove the existence of a special point \( G \) that creates a segment congruent to segment \( DF \).
2. There exists a point \( H \) such that \( H-A-C \). (justify) ____________________________________________________________

3. There exists a point \( G \) on ray(\( AH \)) such that \( AG = DF \). (Draw a picture.) (justify) ________________________________________________________________

Show that the resulting angle is congruent to \( \angle EDF \).
4. Angles \( \angle BAG \) and \( \angle BAC \) are supplemental. (justify) ____________________________________________________________

5. \( m(\angle BAG) = 90 \). (justify) ______________________________________________________________________________________

6. \( \angle BAG \cong \angle EDF \). (justify) ______________________________________________________________________________________

Show that the resulting triangle is congruent to \( \triangle DEF \).
7. \( \triangle ABG \cong \triangle DEF \). (justify) ______________________________________________________________________________________
Show that the resulting triangle is congruent to ΔABC.

8. BG = EF. (justify) ______________________________________

9. BG = BC (justify) ______________________________________

10. ∠ABG ≅ ∠ABC (justify) ________________________________

11. ΔABG ≅ ΔABC (justify) ________________________________

Conclusion

12. ΔDEF ≅ ΔABC (justify) ________________________________

End of Proof
Theorem 1.24 In Euclidean Geometry, in any triangle, the measure of an exterior angle in a triangle is equal to the sum of the measures of its two remote interior angles.

Proof:

1. In Euclidean Geometry, let $\triangle ABC$ be given, and let $D$ be a point such that $B-C-D$. With this configuration, $\angle ACD$ is an exterior angle and $\angle ABC$ and $\angle BAC$ are its remote interior angles.

2. $m(\angle BCA) + m(\angle ACD) = 180$ (justify): ____________________________

3. $m(\angle BCA) + m(\angle CAB) + m(\angle ABC) = 180$ (justify): ______________

4. $m(\angle ACD) = m(\angle CAB) + m(\angle ABC)$ (justify): ________________________

End of Proof
Remember that a parallelogram is defined to be a quadrilateral in which each pair of opposite sides is parallel. The goal for today is to prove the following Theorem about Parallelograms.

*Theorem (1.26 & 1.27 combined)* In Euclidean Geometry, given any convex quadrilateral, the following statements are equivalent (TFAE):

1. The quadrilateral is a parallelogram.
2. Each diagonal divides the quadrilateral into two congruent triangles.
3. Each pair of opposite sides is congruent.
4. The diagonals bisect each other.
5. Each pair of opposite angles is congruent.
6. One pair of opposite sides is both congruent and parallel.

There are many different ways to prove this. For clarity, you should clearly identify parts of your proof with letters. For example, you could write “Proof Part C: Prove that (5) $\Rightarrow$ (2).” Draw arrows connecting the numbers below to indicate which implications you have proven, and next to each arrow, write the letter of the proof part where you prove that arrow.

(1) \hspace{1cm} (2)

(6) \hspace{1cm} (3)

(5) \hspace{1cm} (4)
Here is a theorem that combines a simplified version of Proposition 2.3 and its converse.

**Theorem:** In a circle with chord $AB$ and diameter $CD$, the following statements are equivalent:

1. $CD$ is perpendicular to $AB$.
2. $CD$ bisects $AB$.

Prove the Theorem without using any of Propositions, Theorems, or other results of Chapter 2.

Hint: Let $O$ be the center of the circle and let $M$ be the point of intersection of chord $AB$ and diameter $CD$. Draw radial segments $OA$ and $OB$. Analyze triangles.
The following theorem is famous. Justify the steps in the proof. Make drawings where indicated.

**Theorem:** In Neutral Geometry, the following are equivalent:
1. Line L is perpendicular to segment AP at point P.
2. Line L intersects Circle(A,AP) exactly once, at point P.

Proof part 1: Show that (1) $\implies$ (2)
1) Suppose that Line L is perpendicular to segment AP at point P.
2) Line L intersects Circle(A,AP) at point P. (justification: ___________ __________________________)
3) Assume that line L intersects Circle(A,AP) at another point Q. (assumption) (make a drawing)

4) Segment AP is congruent to segment AQ (justification: ___________ __________________________)
5) Angle AQP is congruent to angle APQ (justification: ___________ __________________________)
6) Lines AQ and AP are parallel (justification: ___________ __________________________)
7) Step 6 contradicts something. (what does it contradict? ___________ __________________________)

So our assumption in step (3) must be wrong. Conclude that Line L intersects Circle(A,AP) exactly once, at point P.

End of proof part 1
Proof part 2: Show that (2) $\Rightarrow$ (1)

1) Suppose that Line L intersects Circle(A,AP) exactly once, at point P.
2) Assume that line L is not perpendicular to segment AP at point P. (assumption)
3) There exists a line M that passes through point A that is perpendicular to line L. (justification: ____________________________)
4) The point of intersection of lines M and L cannot be point P (why not? ____________________________), so we can give the intersection the label Q. So line M is line AQ. (make a drawing)
5) Observe that Angle AQP measures 90 degrees. (justification: ________)
6) There exists a point R such that P-Q-R and such that QP = QR (justification: ____________________________)
7) Angle AQR measures 90 degrees (justification: ____________________________)
8) Segment AQ is congruent to itself. (justification: ____________________________)
9) Triangle AQP is congruent to triangle AQR (justification: ____________________________)
10) Segment AP is congruent to segment AR (justification: ____________________________)
11) Point R is on Circle(A,AP) (justification: ____________________________)
12) Line L intersects Circle(A,AP) at point R. (justification: ____________________________)
13) Step 12 contradicts something. (What does it contradict? __________)

Therefore, our assumption in step 2 was wrong. It cannot be that line L is not perpendicular to segment AP at point P. Conclude line L is perpendicular to segment AP at point P.

End of proof
Prove the following

*Theorem 2.6 If point A is in the exterior of a circle, and line(A,P) and line(A,Q) are tangent to the circle at points P and Q, then AP = AQ.*

Hint: Let O be the center of the circle. Draw segments OP and OQ.
Prove following theorem

*Theorem (angles formed by two lines tangent to a circle)* If point $P$ is in the exterior of a circle, and line$(P,A)$ and line$(P,B)$ are tangent to the circle at points $A$ and $B$, then

$$m(\angle APB) = \frac{m(\text{major arc } AB) - m(\text{minor arc } AB)}{2}.$$ 

Hints:

- Let $O$ be the center of the circle and consider quadrilateral$(P,A,O,B)$.
- Let $\alpha = m(\angle APB)$ and let $\beta = m(\angle AOB)$. Express $\alpha$ in terms of $\beta$.
- Helpful bit of algebra: note that $180 - \beta = \frac{(360 - \beta) - \beta}{2}$. 

In class, I introduced the following Theorem

*Theorem 2.10* Given quadrilateral $ABCD$, the following are equivalent:

1. A tangent circle can be drawn inside $\text{quad}(ABCD)$
2. $AB + CD = AD + BC$

Prove that (1) $\implies$ (2). (The proof that (2) $\implies$ (1) will be in Homework 6.)

Hints:

- Suppose that (1) is true. Let point $O$ be the center of the tangent circle. Let points $P, Q, R, S$ be the points where the circle touches sides $AB, BC, CD, DA$. (Draw a picture)
- By Theorem 2.6. $AS = AP$. Let $x$ be the common length.
- By Theorem 2.6. $BP = BQ$. Let $y$ be the common length.
- By Theorem 2.6. $CQ = CR$. Let $z$ be the common length.
- By Theorem 2.6. $DR = DS$. Let $w$ be the common length.
Prove the following theorem:

*Theorem: If a tangent circle exists inside a parallelogram, then the parallelogram is a rhombus*
(Textbook Exercise 2.2#2.)

Trapezoid(A,B,C,D) with base AB and top CD has a circle inscribed in it. [Draw a picture, labeling the vertices A,B,C,D by putting A in the lower left and then going counter-clockwise .]

The center of the circle is point O, and points P and Q are the points where the circle touches the top and the base.

(a) Show that points O, P, Q are collinear.
(b) Show that the diameter of the circle equals the height of the trapezoid.
(c) Show that m(∠COB) = 90°.
Class Drill 2.2.06 (done in class on Friday, February 19, 2010)

(Textbook Exercise 2.2#11.)

Trapezoid\( (A,B,C,D) \) with base \( AB \) and top \( CD \) has a circle inscribed in it. The center of the circle is point \( O \).

Points \( M \) and \( N \) are the midpoints of sides \( AD \) and \( BC \), respectively.

[Draw a picture, labeling the vertices \( A,B,C,D \) by putting \( A \) in the lower left and then going counter-clockwise .]

(a) Show that the midsegment \( MN \) contains point \( O \).

Hint: There are several ways to prove this. One is to show that the distance from \( M \) to line\((A,B)\) is the same as the distance from \( M \) to line\((C,D)\). To justify this, construct a perpendicular from \( M \) to the bases of the trapezoid and prove that the two right triangles created are congruent. Hence, each distance is half the height of the trapezoid. Similarly, show that the distance from \( N \) to line\((A,B)\) is the same as the distance from \( N \) to line\((C,D)\).

(b) Show that \( OM = MA \) and \( ON = NB \).

Hints:
- Show that \( \angle NOB \cong \angle ABO \).
- Show that \( \angle ABO \cong \angle NBO \).
(Textbook Exercise 2.2#12.)

Trapezoid(A, B, C, D) with base AB and top CD has a circle inscribed in it. Points M and N are the midpoints of sides AD and BC, respectively. Point P is a point on midsegment MN such that PM = MA and PN = NB. [Draw a picture, labeling the vertices A, B, C, D by putting A in the lower left and then going counter-clockwise.]

Prove that a tangent circle exists, centered at point P.

Hint:
Show that ∠PAM ≅ ∠PAB.
Show that ∠PBN ≅ ∠PBA.
Show that ∠PCN ≅ ∠PCD.
Make a conclusion about point P.
Textbook Exercise 2.2#19 is about Miquel’s Theorem:

If A, B, C are three non-collinear points, and D, E, F are points on the segments AC, AB, BC, respectively, then the three circles Circle(A,D,E), Circle(D,F,C), Circle(E,B,F) intersect at a point P

Justify the steps in the following proof:

1) Circle(A,D,E) and Circle(B,E,F) intersect, because they both contain E. There are two cases:

In Case 1, Circle(A,D,E) and Circle(B,E,F) are tangent to each other at point E, and so they don't intersect at any other points (In this case, my goal is to show that point E is also on Circle(C,D,F). I will show that by proving that quad(CDEF) is cyclic.)

2) Let line L be the line that is tangent to both circles at point E. 
   Remark: line L intersects line AB (justify: _____________________________
   _____________________________),
   but line L need not be perpendicular to line(A,B).

3) Let H be a point on L that is on the same side of line(A,B) as points D and F.
4) Observe that \( \angle DEH \cong \angle A \) (justify: _____________________________
   _____________________________)

5) Also observe that \( \angle HEF \cong \angle B \) (justify: ______________
   _____________________________).

6) Therefore \( m(\angle DEF) = m(\angle DEH) + m(\angle HEF) = m(\angle A) + m(\angle B) = 180 - m(\angle C) \).

7) This tells us that quad(CDEF) is cyclic. That is, there exists a circle that passes through all four points C,D,E,F.

8) But that circle must be Circle(C,D,F). (justify: _____________________________)

   Therefore, point E lies on Circle(C,D,F).

9) Conclude that the three circles are concurrent at point E.
In Case 2, Circle(A,D,E) and Circle(B,E,F) are not tangent to each other at point E, so they must intersect at one other point that we can call P. (In this case, my goal is to show that point P is also on Circle(C,D,F). I will show that by proving that \(quad(CDPF)\) is cyclic.)

10) Observe that points A,E,P,D all lie on Circle(A,D,E). That tells us that \(quad(AEPD)\) is cyclic.
11) Therefore, \(m(\angle DPE) = \)______________________________.
12) Observe that points B,F,P,E all lie on Circle(B,E,F). That tells us that \(quad(BFPE)\) is cyclic.
13) Therefore, \(m(\angle EPF) = \)______________________________.
14) We know that \(360 = m(\angle DPE) + m(\angle EPF) + m(\angle DPF)\)
15) Substituting, we get \(360 = \)______________________________
16) Therefore, \(m(\angle DPF) = m(\angle A) + m(\angle B) = \)______________________________
17) This tells us that \(\)______________________________.

That is, there exists a circle that passes through all four points C,D,P,F.
18) But that circle must be Circle(C,D,F), (justify: \(\)______________________________)

Therefore, point P lies on Circle(C,D,F).
19) Conclude that the three circles are concurrent at point P.

Wrap-up
20) We see that in either case, the three circles are concurrent.
End of proof
Explain how to replace fence(AB_{1}B_{2}C) with a single straight fence such that Region II still has the same area.
Find the simplest possible expression for the area of the trapezoid in terms of $a$ and $b$. (Do not use trigonometry.)
Class Drill 3.2.01 (Done in class on Tuesday, March 9, 2010)

Given the segment of length $a$ shown below, construct a segment of length $a\sqrt{34}$ using the "first approach". That is, using the equation $3^2 + 5^2 = 34$. 

\[ a \]

\[
\text{\textbullet} \\
\text{\textbullet}
\]

$\overline{a}$
Given the segment of length $a$ shown below, construct a segment of length $a\sqrt{11}$ using the “second approach”. That is, using the equation

$$11 + 5^2 = 6^2.$$
Class Drill 2.3.01 (Done in class on Tuesday, March 9, 2010)

Given the segment of length $c$ and the angle $\alpha$ shown below:

\[ \text{Construct a triangle } \triangle ABC \text{ such that} \]

- $AB = c$.
- $\angle ACB = \alpha$.

In class, I said that we would refer to this as “construction *.”
Given the segments of length $c$ and $b$ and the angle $\alpha$ shown below:

Construct a triangle $\triangle ABC$ such that

- $AB = c$.
- $\angle ACB = \alpha$.
- $AC = b$. 