Math 163A Handout 5: Some Functions and their Derivatives from Section 5-4

Section 5-4 of the textbook is about techniques for sketching graphs of functions. The problems are to be done using the Graphing Strategy on page 321. This strategy involves analyzing a function \( f \) and its derivatives \( f' \) and \( f'' \). These are all very long problems.

One of the things that contributes to the length of the problems is the task of finding the derivatives and factoring them. This is a difficult task, one that ultimately must be mastered. But I feel that the focus in Section 5-4 should be on the curve sketching, and that a student’s difficulty with derivatives and factoring should not be an obstacle to their being able practice the graphing strategy.

For that reason, I have prepared this handout of functions and their derivatives. The functions are drawn from the suggested problem list for Section 5-4 of the textbook, and also from the assigned problems for Math 163A Section A04.

Throughout this handout, a \{factor in curly brackets\} is one whose sign never changes, while a \[factor in square brackets\] is one that can be positive or zero, but never negative.

(a) \( f(x) = \frac{x+3}{x-3} \) (rational function from suggested exercise 5-4#11) (see Example 1, p. 321)

\[
f'(x) = -\frac{6}{(x-3)^2} = \{-6\} \cdot \frac{1}{(x-3)^2}
\]

\[
f''(x) = \frac{12}{(x-3)^3} = \{12\} \cdot \frac{1}{(x-3)^3}
\]

(b) \( f(x) = \frac{1}{x^2+1} \) (rational function from suggested exercise 5-4#25) (done in class Fri. Nov 7)

\[
f'(x) = -\frac{2x}{(x^2+1)^2} = \left\{-\frac{2}{(x^2+1)^2}\right\} \cdot x
\]

\[
f''(x) = \frac{2(3x^2-1)}{(x^2+1)^3} = \frac{6\left(x^2 - \frac{1}{3}\right)}{(x^2+1)^3} = \left\{\frac{6}{(x^2+1)^3}\right\} \left(x + \frac{1}{\sqrt{3}}\right) \left(x - \frac{1}{\sqrt{3}}\right)
\]

(c) \( f(x) = \frac{x}{x^2+1} \) (rational function from assigned problem H6#5) (see homework solutions)

\[
f'(x) = \frac{(x+1)(x-1)}{(x^2+1)^2} = \left\{-\frac{1}{(x^2+1)^2}\right\} \cdot (x+1)(x-1)
\]

\[
f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3} = \left\{\frac{2}{(x^2+1)^3}\right\} \cdot x \left(x + \sqrt{3}\right) \left(x - \sqrt{3}\right)
\]
(d) \[ f(x) = \frac{1}{x^2 - 1} = \frac{1}{(x+1)(x-1)} \] (similar to rational function from suggested exercise 5-4#24)
\[ f'(x) = -\frac{2x}{(x^2 - 1)^2} = \{-2\} \cdot \frac{x}{[(x+1)^2][(x-1)^2]} \]
\[ f''(x) = \frac{2(3x^2 + 1)}{(x^2 - 1)^3} = \left\{2\left(\frac{3x^2 + 1}{x}ight)\right\} \cdot \frac{1}{(x+1)^3(x-1)^3} \]

(e) \[ f(x) = \frac{x}{x^2 - 1} = \frac{x}{(x+1)(x-1)} \] (similar to rational function from suggested exercise 5-4#23)
\[ f'(x) = -\frac{(x^2 + 1)}{(x^2 - 1)^2} = \{-\left(\frac{x^2 + 1}{x}\right)\} \cdot \frac{1}{[(x+1)^2][(x-1)^2]} \]
\[ f''(x) = \frac{2x(x^2 + 3)}{(x^2 - 1)^3} = \left\{2\left(\frac{x^2 + 3}{x}\right)\right\} \cdot \frac{x}{(x+1)^3(x-1)^3} \]

(f) \[ f(x) = \frac{1}{x^2 - 4} = \frac{1}{(x+2)(x-2)} \] (rational function from suggested exercise 5-4#24)
\[ f'(x) = -\frac{2x}{(x^2 - 4)^2} = \{-2\} \cdot \frac{x}{[(x+2)^2][(x-2)^2]} \]
\[ f''(x) = \frac{2(3x^2 + 4)}{(x^2 - 4)^3} = \left\{2\left(\frac{3x^2 + 4}{x}\right)\right\} \cdot \frac{1}{(x+2)^3(x-2)^3} \]

(g) \[ f(x) = \frac{x}{x^2 - 4} = \frac{x}{(x+2)(x-2)} \] (rational function from suggested exercise 5-4#23)
\[ f'(x) = -\frac{(x^2 + 4)}{(x^2 - 4)^2} = \{-\left(\frac{x^2 + 4}{x}\right)\} \cdot \frac{1}{[(x+2)^2][(x-2)^2]} \]
\[ f''(x) = \frac{2x(x^2 + 12)}{(x^2 - 4)^3} = \left\{2\left(\frac{x^2 + 12}{x}\right)\right\} \cdot \frac{x}{(x+2)^3(x-2)^3} \]

(h) \[ f(x) = \frac{x}{x^2 - 4} = \frac{x}{(x+2)(x-2)} \] (rational function from suggested exercise 5-4#23)
\[ f'(x) = -\frac{(x^2 + 4)}{(x^2 - 4)^2} = \{-\left(\frac{x^2 + 4}{x}\right)\} \cdot \frac{1}{[(x+2)^2][(x-2)^2]} \]
\[ f''(x) = \frac{2x(x^2 + 12)}{(x^2 - 4)^3} = \left\{2\left(\frac{x^2 + 12}{x}\right)\right\} \cdot \frac{x}{(x+2)^3(x-2)^3} \]
(i) \[ f(x) = xe^{-(x)} = \left\{ e^{-(x)} \right\} \cdot x \text{ (similar to function from suggested exercise 5-4#17) (see Example 3, p. 324)} \]
\[ f'(x) = -(x-1)e^{-(x)} = \left\{ -e^{-(x)} \right\} \cdot (x-1) \]
\[ f''(x) = (x-2)e^{-(x)} = \left\{ e^{-(x)} \right\} \cdot (x-2) \]

(j) \[ f(x) = 5xe^{-(0.2x)} = \left\{ 5e^{-\left(\frac{1}{5}x\right)} \right\} \cdot x \text{ (function from suggested exercise 5-4#17) (see Example 3, p. 324)} \]
\[ f'(x) = -(x-5)e^{-\left(\frac{1}{5}x\right)} = \left\{ -e^{-\left(\frac{1}{5}x\right)} \right\} \cdot (x-5) \]
\[ f''(x) = \left(\frac{1}{5}\right)(x-10)e^{-\left(\frac{1}{5}x\right)} = \left\{ \left(\frac{1}{5}\right)e^{-\left(\frac{1}{5}x\right)} \right\} \cdot (x-10) \]

(k) \[ f(x) = (3-x)e^{(x)} = \left\{ -e^{(x)} \right\} \cdot (x-3) \text{ (function from suggested exercise 5-4#41) (see Example 3, p. 324)} \]
\[ f'(x) = -(x-2)e^{(x)} = \left\{ -e^{(x)} \right\} \cdot (x-2) \]
\[ f''(x) = -(x-1)e^{(x)} = \left\{ -e^{(x)} \right\} \cdot (x-1) \]

(l) \[ f(x) = e^{-x^2} = \left\{ e^{-x^2} \right\} \text{ (function from assigned problem H5#4) (see solutions)} \]
\[ f'(x) = -2xe^{-x^2} = \left\{ -2e^{-x^2} \right\} \cdot x \]
\[ f''(x) = 2(2x^2-1)e^{-x^2} = 4\left(x^2 - \frac{1}{2}\right)e^{-x^2} = \left\{ 4e^{-x^2} \right\} \cdot \left(x + \left(\frac{1}{\sqrt{2}}\right)\right)\left(x - \left(\frac{1}{\sqrt{2}}\right)\right) \]

(m) \[ f(x) = e^{-0.5x^2} = \left\{ e^{-\left(\frac{1}{2}x^2\right)} \right\} \text{ (function from suggested problem 5-4#43) (see solutions to H5#4)} \]
\[ f'(x) = -xe^{-\left(\frac{1}{2}x^2\right)} = \left\{ -e^{-\left(\frac{1}{2}x^2\right)} \right\} \cdot x \]
\[ f''(x) = \left(x^2 - 1\right)e^{-\left(\frac{1}{2}x^2\right)} = \left\{ e^{-\left(\frac{1}{2}x^2\right)} \right\} \cdot (x+1)(x-1) \]