[1] (problem 4.1#7, done in class)
In triangle(ABC), point D is the midpoint of side(AB) and point E is the midpoint of side(AC).
(a) Using similarity, prove the Triangle Midsegment Theorem:
segment(BC) \parallel \text{segment}(DE) \text{ and } BC = 2DE.

Solution:
Show that triangle(BAC) is similar to triangle(DAE)
• Observe that angle(BAC) is congruent to itself.
• Observe that BA/DA = 2 = CA/EA, because points D and E are midpoints.
• Therefore, triangle(BAC) \sim triangle(DAE) by ASA similarity.
Use the similarity result to show that BC = 2DE.
• We know that BC/DE = 2 because similar triangles have corresponding sides that are proportional.
• Conclude that BC = 2DE.
Use the similarity result to show that segment(BC) is parallel to segment(DE)
• We know that angle(ABC) is congruent to angle(ADE) because similar triangles have corresponding angles that are congruent.
• Therefore, line(BC) is parallel to line(DE) by the converse of the corresponding angle theorem.
End of proof.

(b) Medians BE and CD meet at point F. Using similarity, prove that BF = 2FE and CF = 2FD.

Solution
Show that triangle(CBF) is similar to triangle(DEF)
• Angle(CBF) is congruent to angle(DEF). (We know from part (a) that BC \parallel DE. Angle(CBF) and angle(DEF) are alternate interior angles. The Converse of the Alternate Interior Angle Theorem says that they are congruent.)
• Angle(BFC) is congruent to (EFD) because they are vertical angles.
• Therefore, triangle(CBF) \sim triangle(DEF) by AA similarity.
Use the similarity result
• BF/EF = CF/DF = BC/DE because similar triangles have corresponding sides that are proportional.
• But we know from part (a) that BC/DE = 2.
• Therefore, BF/EF = 2 and CF/DF = 2. Therefore, BF = 2EF and CF = 2DF.
End of proof.

[2] (suggested problem 4.2#2)
In triangle(ABC), angle C is a right angle. D is a point on side(BC) such that ray(AD) bisects angle(A). Which segment is longer: BD or CD? Prove your answer.

Solution
• BD/CD = BA/CA by Theorem 4.7 (side splitting) applied to triangle(ABC) and bisector ray(AD).
• But BA > CA because BA is the hypotenuse and CA is a leg in right triangle(ABC).
• Therefore, BA/CA > 1.
• Therefore, BD/CD > 1.
• Therefore, BD > CD.
[3] (part of assigned problem 4.3#5)
A hexagon #1 has sides 1 unit long. A hexagon #2 is inscribed by joining the midpoints of the sides of the first hexagon. What is ratio area(hexagon #1) / area(hexagon #2). Show your work and explain.

Solution:
Consider one of the six equilateral triangular wedges that make up hexagon #1. This wedge will have three sides of length 1. One of the sides of this triangle is a side of hexagon #1. The other two sides of the triangle are radial segments of hexagon #1. An altitude segment of this triangle will be a radial segment of hexagon #1. But the altitude segment will have length = sqrt(3)/2 because the altitude segment is the long leg of a 30-60-90 triangle that has hypotenuse of length 1. See the figure below.

So (length of radius of hex #1)/(length of radius of hex #1) = 1/(sqrt(3)/2) = 2/sqrt(3).
Therefore, (area of hex #1)/(area of hex #2) = (2/sqrt(3))^2 = 4/3

[4] (assigned problem 4.4#1)
In triangle(ABC), m(angle(A)) = 36, and m(angle(B)) = m(angle(C)) = 72. Prove that the ratio of lengths BA/BC is equal to $\phi$, the golden ratio.

Hints:
- Let BC = 1. Then your job is to show that BA is equal to $\phi$.
- Let D be a point on AB such that ray(CD) bisects angle(C). Use similarity and side splitting.

Solution (This solution is slightly different from the solution to problem 4.4#1 that I included in the Homework #3 Solutions. I think this solution may be easier.)
(Draw a picture as you read this proof)

Setup
- Suppose that triangle(ABC) has a 36 degree angle at A and 72 degree angles at B and C. Suppose further that BC = 1.
- Let BA = x. Our goal is to find BA/BC. Because CB = 1, we can find the ratio by just finding x.
- Notice that because triangle(ABC) is isosceles, CA = x as well.
- Let D be the point on side(BA) such that ray(CD) bisects angle(C).

Use the Side splitting Theorem
- Apply Theorem 4.7 (Side Splitting) to triangle(ABC) with bisector CD. Result: DA/DB = CA/CB = x.
- Since DA/DB = x, we can multiply both sides by DB to obtain the new equation DA = xDB.

Use segment addition
- From segment addition, we know that DB + DA = x.
- Substituting in DA = xDB, we obtain the new equation DB + xDB = x.
- Factoring this equation, we obtain the new equation DB(1 + x) = x.
- Dividing, we obtain the new equation DB = x/(1 + x). Call this equation *
Use fact that the small triangle is similar to the large one
• Note that measure(angle(BCD)) = 36 because ray CD bisects angle(C).
• Observe that triangle(ABC) is similar to triangle(CDB) by AA similarity.
• Therefore CA/CB = CB/BD. That is x = 1/BD.
• This tells us that BD = 1/x. Call this equation **

Combine results
• Combining equation* and equation **, we obtain the new equation x/(1 + x) = 1/x.
• Multiplying both sides by x(1 + x) and cancelling, we obtain the new equation x^2 = x + 1.
• Subtracting 1 + x from both sides of this equation, we obtain the new equation x^2 - x - 1 = 0.
• We see that x satisfies the same equation that φ satisfies. We also know that x is positive. Therefore, x must be the same number as φ.

[5] (suggested problem 4.5#3)
In each picture, the square has sides of length 1. Find the shaded areas. Show your work clearly.

Solution:
(a) Shaded area = area of square - area of 1 circle of radius r = 1/2
= 1 - π(1/2)^2
= 1 - π/4

(b) Shaded area = area of square - area of 4 circles of radius r = 1/4
= 1 - 4π(1/4)^2
= 1 - π/4

(c) Shaded area = area of square - area of 9 circles of radius r = 1/6
= 1 - 9π(1/6)^2
= 1 - 9π/36
= 1 - π/4

(d) Shaded area = area of square - area of n^2 circles of radius r = 1/(2n)
= 1 - n^2π(1/(2n))^2
= 1 - n^2π/(4n^2)
= 1 - π/4