Math 163A Handout 4: Six Step Method for Graphing Rational Functions without Calculus

**Step 1:** If \( x = 0 \) is in the domain, find \( f(0) \).

**Step 2:** Check for symmetries.

**Step 3:** Determine the end behavior (horizontal asymptote? slant asymptote? power function?) by deciding which of the following three cases applies.

- **case 1:** degree of numerator < degree of denominator
  
  In this case, the end behavior will resemble \( y = \frac{1}{x^m} \). So the line \( y = 0 \) will be a horizontal asymptote.

- **case 2:** degree of numerator = degree of denominator
  
  In this case, the end behavior will resemble \( y = \frac{a x^m}{b x^m} = \frac{a}{b} \). So the line \( y = \frac{a}{b} \) will be a horizontal asymptote.

- **case 3:** degree of numerator > degree of denominator
  
  **case 3a:** degree of numerator = 1 + degree of denominator
  
  In this case, the the end behavior will resemble \( y = \frac{ax^{m+1}}{bx^m} = \frac{a}{b} x \). So, the line \( y = \frac{a}{b} x \) will be a slant asymptote.

  **case 3b:** degree of numerator \( \geq 2 + \) degree of denominator
  
  In this case, the end behavior will resemble \( y = x^m \), for some integer \( m \geq 2 \).

**Step 4:** Make a sign chart for \( f \). To do this, factor the function (both numerator and denominator). This will enable you to determine the domain of the function, and it will also tell you the important \( x \)-values for \( f \). (If a linear factor \( (x-r) \) appears in the factorization, then the number \( r \) is an important \( x \)-value.) Then, put all the important \( x \)-values on a number line. In each region and at each important \( x \)-value, determine whether \( f \) is positive, negative, zero, or undefined.

**Step 5:** Locate vertical asymptotes, holes, and \( x \)-intercepts by examining the linear factors in the factorization. For each linear factor \( (x-r) \) in the factorization, decide which of the following five cases applies.

- **case 1:** The linear factor \( (x-r) \) appears in the numerator but not in the denominator.
  
  In this case, the graph will have an \( x \)-intercept at \( x = r \).

- **case 2:** The linear factor \( (x-r) \) appears in both the numerator and denominator but with a larger exponent in the numerator.
  
  In this case, the graph will cross the \( x \)-axis at \( x = r \), but there will be a hole at the crossing.

- **case 3:** The linear factor \( (x-r) \) appears in both the numerator and in the denominator and with equal exponents.
  
  In this case, the graph will have a hole at \( x = r \).

- **case 4:** The linear factor \( (x-r) \) appears in both the numerator and denominator but with a smaller exponent in the numerator.
  
  In this case, the graph will have a vertical asymptote at \( x = r \).

- **case 5:** The linear factor \( (x-r) \) appears in the denominator only.
  
  In this case, the graph will have a vertical asymptote at \( x = r \).

**Step 6:** Based on the analysis in steps 1 through step 5, sketch the graph of \( f \).