



MATH 340 CCE – Sample Examination

1. Solve the equation: $y' = e^{2x-y}$ $y(0) = \ln \frac{3}{2}$.
2. Find the general solution of: $y' + y \cot x = 2 \cos x$.
3. Find the general solution of: $y'' + y = \cos^2 x$.
4. Find the general solution of: $y'' + 5y' + 4y = x^2$.
5. Prove or disprove that $\{\cos(\ln x), \sin(\ln x)\}$ is a fundamental set of solutions for $x^2 y'' + xy' + y = 0$ on $(0, \infty)$.
6. $y(x) = -e^{-2x} \sin(e^x)$ is a solution to $y'' + 3y' + 2y = \sin(e^x)$.
Find the solution to this equation which satisfies $y(0) = 1, y'(0) = 0$.
7. Use the power series method to find the general solution of $y' - xy = 0$.
8. The differential equation $\frac{dy}{dx} = \frac{3y}{3x + y}$ can be solved by use of the substitution $y = vx$.
Use this substitution to solve the equation.
9. (a) The auxiliary equation to a certain linear constant coefficient homogeneous differential equation has the following roots:
 $0, 0, 1, -1, i, -i, i, -i, 2 + 3i, 2 - 3i$
What is the general solution to the differential equation?
(b) What is the annihilator of the following function: $f(x) = e^x \sin 2x + e^{-x}$?
10. Solve $y'' + 4y = f(t)$ where $y(0) = 0, y'(0) = 0$, and $f(t)$ is given by
$$f(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & t \geq 1 \end{cases}$$
11. Solve $x_1' - x_2 = 0$ $x_1(0) = 1$
 $x_2' + x_1 = e^t$ $x_2(0) = 0$
12. A glass of ice water is placed in a room which is at 70°F . After ten minutes the water in the glass is at 40°F . How long will it take for the water to reach 60°F ? Assume that the rate at which the water temperature changes is proportional to the difference between the room temperature and the water temperature.

SOLUTIONS TO SAMPLE EXAMINATION PROBLEMS

1.

$$\frac{dy}{dx} = e^{2x-y} = e^{2x}e^{-y}$$

$$e^y \frac{dy}{dx} = e^{2x}$$

$$e^y dy = e^{2x} dx \text{ variables are separated}$$

$$e^y = \frac{1}{2}e^{2x} + C$$

$$e^{y(0)} = \frac{1}{2}e^{2 \cdot 0} + C$$

$$e^{\ln \frac{3}{2}} = \frac{1}{2} + C$$

$$\frac{3}{2} = \frac{1}{2} + C$$

$$C = 1$$

$$e^y = \frac{1}{2}e^{2x} + 1$$

$$y = \ln \left(\frac{1}{2}e^{2x} + 1 \right)$$

2. An integrating factor is $e^{\int \cot x \, dx} = e^{\ln \sin x} = \sin x$.

$$y' \sin x + y \cot x \sin x = 2 \cos x \sin x$$

$$y' \sin x + y \cos x = 2 \cos x \sin x$$

$$\frac{d}{dx}(y \sin x) = \frac{d}{dx}(\sin^2 x)$$

$$y \sin x = \sin^2 x + C$$

$$y = \sin x + \frac{C}{\sin x}$$

(continued)

3. We use variation of parameters. $\{\sin x, \cos x\}$ is a fundamental set of solutions for $y'' + y = 0$. To find a particular solution we assume a solution of the form $y_p = u_1 \sin x + u_2 \cos x$. Then $y_p' = u_1' \sin x + u_1 \cos x + u_2' \cos x - u_2 \sin x$. To avoid the appearance of 2nd derivative terms in u_1 and u_2 we impose the following condition:

$$u_1' \sin x + u_2' \cos x = 0 \quad (1)$$

Then

$$\begin{aligned} y_p' &= u_1 \cos x - u_2 \sin x \\ y_p'' &= u_1' \cos x - u_1 \sin x - u_2' \sin x - u_2 \cos x \end{aligned}$$

Substitute these into the given equation to obtain

$$y_p'' + y_p = u_1' \cos x - u_2' \sin x = \cos^2 x \quad (2)$$

We must solve (1) and (2) simultaneously, viz.,

$$\begin{aligned} u_1' \cos x - u_2' \sin x &= \cos^2 x \\ u_1' \sin x + u_2' \cos x &= 0 \end{aligned}$$

Multiply the top equation by $\cos x$, the bottom equation by $\sin x$, and add to obtain

$$\begin{aligned} u_1' &= \cos^3 x \\ u_1' &= (1 - \sin^2 x) \cos x = \cos x - \sin^2 x \cos x \\ u_1 &= \sin x - \frac{\sin^3 x}{3} \end{aligned} \quad (3)$$

Substitute (3) into (1) to obtain

$$\begin{aligned} \cos^3 x \sin x + u_2' \cos x &= 0 \\ u_2' &= -\cos^2 x \sin x \\ u_2 &= \frac{\cos^3 x}{3} \end{aligned}$$

(continued)

Therefore,

$$\begin{aligned}y_p &= \sin^2 x - \frac{\sin^4 x}{3} + \frac{\cos^4 x}{3} \\&= \sin^2 x + \frac{1}{3}(\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x) \\&= \sin^2 x + \frac{1}{3}(\cos^2 x - \sin^2 x) \\&= \frac{1}{3}\cos^2 x + \frac{2}{3}\sin^2 x = \frac{1}{3}(1 + \sin^2 x) \\y &= C_1 \sin x + C_2 \cos x + \frac{1}{3} + \frac{1}{3}\sin^2 x\end{aligned}$$

4. $y'' + 5y' + 4y = x^2$ is easily solved by the method of undetermined coefficients. $(D^2 + 5D + 4)y = x^2$ and D^3 annihilates x^2 , so $D^3(D^2 + 5D + 4)y = 0$. For a particular solution we use the factors not in the homogeneous equation to get $y_p = A + Bx + Cx^2$. Substitute into the given differential equation to get

$$\begin{aligned}2C + 5(B + 2Cx) + 4(A + Bx + Cx^2) &= x^2 \\4Cx^2 + (10C + 4B)x + (2C + 5B + 4A) &= x^2\end{aligned}$$

$$\text{Then } 4C = 1, \quad C = \frac{1}{4}$$

$$10C + 4B = 0, \quad \frac{10}{4} + 4B = 0, \quad B = -\frac{5}{8}$$

$$2C + 5B + 4A = 0, \quad \frac{1}{2} - \frac{25}{8} + 4A = 0, \quad A = \frac{21}{32}$$

$$y_p = \frac{21}{32} - \frac{5}{8}x + \frac{1}{4}x^2$$

The general solution is

$$y = C_1 e^{-4x} + C_2 e^{-x} + \frac{1}{4} - \frac{5}{8}x + \frac{21}{32}$$

where $\{e^{-4x}, e^{-x}\}$ is a fundamental set of solutions for the homogeneous equation.

(continued)

5.

$$y_1 = \cos \ln x$$

$$y_2 = \sin \ln x$$

$$y_1' = -\frac{\sin \ln x}{x}$$

$$y_2' = \frac{\cos \ln x}{x}$$

$$y_1'' = \frac{-\cos \ln x + \sin \ln x}{x^2}$$

$$y_2'' = \frac{-\sin \ln x - \cos \ln x}{x^2}$$

$$x^2 y_1'' + x y_1' + y_1 = x^2 \left[\frac{-\cos \ln x + \sin \ln x}{x^2} \right] + x \left(\frac{-\sin \ln x}{x} \right) + \cos \ln x = 0$$

$$x^2 y_2'' + x y_2' + y_2 = x^2 \left[\frac{-\sin \ln x - \cos \ln x}{x^2} \right] + x \left(\frac{\cos \ln x}{x} \right) + \sin \ln x = 0$$

Thus, y_1 and y_2 are solutions. Next, to show that these solutions are linearly independent we compute their Wronskian.

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} \frac{\cos \ln x}{x} & \frac{\sin \ln x}{x} \\ -\frac{\sin \ln x}{x} & \frac{\cos \ln x}{x} \end{vmatrix} \\ &= \frac{\cos^2 \ln x}{x} + \frac{\sin^2 \ln x}{x} = \frac{1}{x} > 0 \text{ on } (0, \infty) \end{aligned}$$

Thus, the given functions form a fundamental set.

6. The homogeneous equation has $\{e^{-x}, e^{-2x}\}$ as a fundamental set. Hence the general solution is

$$y = C_1 e^{-x} + C_2 e^{-2x} - e^{-2x} \sin(e^x)$$

$$y' = -C_1 e^{-x} - 2C_2 e^{-2x} + 2e^{-2x} \sin(e^x) - e^{-x} \cos(e^x)$$

At $x = 0$

$$y(0) = 1 = C_1 + C_2 - \sin 1$$

$$y'(0) = 0 = -C_1 - 2C_2 + 2\sin 1 - \cos 1$$

We must solve the following equations simultaneously:

$$C_1 + C_2 = 1 + \sin 1$$

$$C_1 + 2C_2 = 2\sin 1 - \cos 1$$

We obtain $C_1 = 2 + \cos 1$ and $C_2 = \sin 1 - \cos 1 - 1$

The desired solution is $y = e^{-x}(2 + \cos 1) + e^{-2x}(\sin 1 - \cos 1 - 1) - e^{-2x} \sin(e^x)$

7. Assume that $y = \sum_{n=0}^{\infty} c_n x^n$.

$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$ and substitution into the DE yields

$$\sum_{n=1}^{\infty} n c_n x^{n-1} - x \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

Let $j = n - 1$ in the first sum and let $j = n + 1$ in the second sum. Then

$$\sum_{j=0}^{\infty} (j+1)c_{j+1}x^j - \sum_{j=1}^{\infty} c_{j-1}x^j = 0$$

Separate out the $j = 0$ term in the first sum and combine the two remaining sums to get

$$c_1 + \sum_{j=1}^{\infty} [(j+1)c_{j+1} - c_{j-1}]x^j = 0$$

c_0 is arbitrary, $c_1 = 0$ as there is no nonzero constant term on the right side of the equation, and we must have $(j+1)c_{j+1} - c_{j-1} = 0$, $j = 1, 2, 3, \dots$ or

$$c_{j+1} = \frac{c_{j-1}}{j+1}, j = 1, 2, 3, \dots$$

$$j = 1 \quad c_2 = \frac{c_0}{2}$$

$$j = 2 \quad c_3 = \frac{c_1}{3} = 0 \text{ since } c_1 = 0$$

$$j = 3 \quad c_4 = \frac{c_2}{4} = \frac{c_0}{2 \cdot 4}$$

$$j = 4 \quad c_5 = \frac{c_3}{5} = 0$$

$$j = 5 \quad c_6 = \frac{c_4}{6} = \frac{c_0}{2 \cdot 4 \cdot 6}$$

(continued)

Thus we can see that $c_{2n} = \frac{c_0}{n!2^n}$ and $c_{2n+1} = 0$.

Therefore, $y = \sum_{n=0}^{\infty} \frac{c_0}{n!2^n} x^{2n}$.

8.

$$y' = v/x + v$$

$$\frac{dv}{dx} x + v = \frac{3vx}{3x + vx} = \frac{3v}{3 + v}$$

$$x \frac{dv}{dx} = \frac{3v}{3 + v} - v = \frac{-v^2}{3 + v}$$

$$\frac{(3 + v)}{v^2} dv = -\frac{dx}{x}$$

$$\frac{3}{v^2} dv + \frac{1}{v} dv = -\frac{dx}{x}$$

$$-\frac{3}{v} + \ln |v| = -\ln |x| + C$$

Let $C = -\ln K$ and use $v = \frac{y}{x}$ to obtain

$$\frac{3x}{y} = \ln \left| \frac{y}{x} \right| + \ln |x| + \ln K = \ln K |y|. \text{ Hence, } x = \frac{1}{3} y \ln K |y|$$

9. (a) $y(x) = C_1 + C_2 x + C_3 e^x + C_4 e^{-x} + C_5 \sin x + C_6 \cos x + C_7 x \sin x + C_8 x \cos x + C_9 e^{2x} \sin 3x + C_{10} e^{2x} \cos 3x$

(b) e^{-x} results if the auxiliary equation has -1 as a root and $e^x \sin 2x$ results if the auxiliary equation has $1 \pm 2i$ as roots. So, $(D + 1)e^{-x} = 0$ and $(D - (1 + 2i))(D - (1 - 2i))e^x \sin 2x = (D^2 - 2D + 5)e^x \sin 2x = 0$. The product of the annihilators is the desired annihilator, viz., $(D + 1)(D^2 - 2D + 5)$.

(continued)

$$10. \quad f(t) = U(t) - U(t - 1) \text{ where } U(t - a) = \begin{cases} 0 & 0 \leq t < a \\ 1 & t \geq a \end{cases}$$

$$L(f) = \frac{1}{s} - \frac{e^{-s}}{s}$$

Transform the given DE and use the given initial conditions to obtain

$$s^2 Y(s) + 4Y(s) = \frac{1}{s}(1 - e^{-s})$$

$$Y(s) = \frac{1}{s(s^2 + 4)} - \frac{e^{-s}}{s(s^2 + 4)}$$

$L^{-1}\left\{\frac{1}{s(s^2 + 4)}\right\}$ may be found by use of partial fractions or from the fact that

$$L\left\{\int_0^t f(x) dx\right\} = \frac{L(f)}{s}$$

We'll use the latter to obtain $L\left\{\int_0^t \frac{1}{2} \sin 2x dx\right\} = \frac{1}{s(s^2 + 4)}$

Thus, $L^{-1}\left\{\frac{1}{s(s^2 + 4)}\right\} = \frac{1}{2} \int_0^t \sin 2x dx = \frac{1}{4} - \frac{1}{4} \cos 2t$ and we have

$$y(t) = \frac{1}{4}(1 - \cos 2t) - \frac{1}{4}(1 - \cos 2(t - 1))U(t - 1)$$

(continued)

11. $sX_1(s) - x_1(0) - X_2(s) = 0$

$$X_1(s) + sX_2(s) - x_2(0) = \frac{1}{s-1}$$

Use the initial conditions to obtain

$$sX_1(s) - X_2(s) = 1$$

$$X_1(s) + sX_2(s) = \frac{1}{s-1}$$

Multiply the first equation by s and add the two equations to obtain

$$(s^2 + 1)X_1(s) = s + \frac{1}{s-1}$$

$$X_1(s) = \frac{s}{s^2 + 1} + \frac{1}{(s-1)(s^2 + 1)}$$

$$X_1(t) = \cos t + L^{-1}\left\{\frac{1}{(s-1)(s^2 + 1)}\right\}$$

From partial fractions we have

$$\frac{1}{(s-1)(s^2 + 1)} = \frac{A}{s-1} + \frac{Bs + C}{s^2 + 1}$$

$$1 = A(s^2 + 1) + (Bs + C)(s - 1)$$

$$1 = (A + B)s^2 + (C - B)s + (A - C)$$

$$A + B = 0 \quad \text{From the first two equations } A + C = 0$$

$$C - B = 0 \quad \text{Solve this one and the third simultaneously to obtain}$$

$$A - C = 1 \quad A = \frac{1}{2} \text{ and } C = -\frac{1}{2}. \text{ Then } B = -\frac{1}{2}$$

(continued)

$$x_1(t) = \cos t + \frac{1}{2}e^t - \frac{1}{2}\cos t - \frac{1}{2}\sin t$$

$$x_1(t) = \frac{1}{2}[e^t + \cos t - \sin t]$$

$$\text{From } x_1' = x_2 \text{ we have } x_2(t) = \frac{1}{2}[e^t - \sin t - \cos t]$$

12. The differential equation which models the phenomena is

$$T'(t) = K(70 - T(t)) \quad T(0) = 32, \quad T(10) = 40$$

$$\frac{dT}{dt} = K(70 - T)$$

$$\frac{dT}{70 - T} = K dt$$

$$\ln(70 - T) = Kt + C$$

When $t = 0$, $T = 32$ and we have $\ln 38 = C$

$$\ln(70 - T) = Kt + \ln 38$$

When $t = 10$, $T = 40$ and we have

$$\ln(70 - 40) = 10K + \ln 38$$

$$\ln 30 - \ln 38 = 10K$$

$$K = \frac{1}{10} \ln\left(\frac{30}{38}\right)$$

$$\ln(70 - T) = \frac{t}{10} \ln\left(\frac{30}{38}\right) + \ln 38$$

(continued)

To find when $T = 60$ we substitute $T = 60$ and solve for t .

$$\ln 10 = \frac{t}{10} \ln \left(\frac{30}{38} \right) + \ln 38$$

$$t = 10 \left(\frac{\ln \frac{10}{38}}{\ln \frac{30}{38}} \right)$$