OHIO UNIVERSITY

MATH 263D CCE — SAMPLE EXAMINATION

You will gain the most benefit from this sample exam if you take it as if it were the supervised examination. Set yourself a time limit of 3 hours and do not use your textbook or notes. Practice writing out all the steps in your solutions to the problems, since you will have to show your work on the problems for the supervised examination.

1. Consider the motion of a point $P(t)$ moving along a helix given by the position vector

$$\vec{r}(t) = 2(\sin t)\vec{i} + 2(\cos t)\vec{j} + t^2\vec{k} = x\vec{i} + y\vec{j} + z\vec{k}$$

where $x = 2\sin t$, $y = 2\cos t$, $z = t^2$, and t measures time.

(a) Compute the velocity vector $\vec{r}\,'(t)$ and the speed $|\vec{r}\,'(t)|$ of the point $P(t)$ at time t, where $t \geq 0$. Does the point $P(t)$ ever stop moving? Explain.

(b) Show that the point $P(t)$ lies on a surface in three-space with Cartesian coordinate equation $x^2 + y^2 = 4$. Identify this surface, and find its equation in cylindrical coordinates.

2. (a) Make the following changes in the given equations below. (Here, (r, θ, z) denote cylindrical coordinates, and (ρ, θ, φ) denote spherical coordinates).

(i) Change $\rho = 5$ to rectangular coordinates.

(ii) Change $\rho = 2\cos\varphi$ to cylindrical coordinates.

(iii) Change $r^2 + z^2 = 9$ to rectangular coordinates.

(b) By inspection, *name* the graph in three-space of each of the following equations in rectangular coordinates.

(1) $x^2 + 5y^2 + z^2 = 25$

(2) $9x^2 - y^2 + 9z^2 = 9$

(3) $\dfrac{x^2}{4} + \dfrac{y^2}{9} = z$

3. (a) Define the partial derivative $\dfrac{\partial f(x, y, z)}{\partial x}$ as a limit.

(b) For $f(x, y, z) = (x + y^2 + z^3)^2$, compute:

(i) $\dfrac{\partial f(x, y, z)}{\partial x}$;

(ii) $\dfrac{\partial^2 f(x, y, z)}{\partial y \partial x}$

(c) For $f(x, u, z) = (x^3 + y^2 + z)^4$, compute $fy(0, 1, 1)$.

4. For $f(x, y) = x^2 + y^2$, find:

(a) The gradient $\vec{\nabla} f(-2, 3)$ at the point $(-2, 3)$.

(b) The directional derivative of $f(x, y)$ at $(-2, 3)$ in the direction of the vector $\vec{a} = 2\vec{i} - 3\vec{j}$.

(c) What is the *maximum rate of change* of $f(x, y)$ at $(-2, 3)$ in the direction at which $f(x, y)$ is increasing most rapidly?

5. Given the surface $z = x^2 + xy + y^2 = f(x, y)$, compute:

(a) The equation of the tangent plane to the surface at the point $(1, 2, 7)$ on the surface.

(b) The rate of change of z with respect to time when $x = 1$, $y = 2$, given that x and y are functions of time, and $\dfrac{dx}{dt} = 2$, $\dfrac{dy}{dt} = 3$.

6. (a) Find all second-order partial derivatives of $u = x \sin y$; that is, find $u_{xx}, u_{yy}, u_{xy}, u_{yx}$

(b) An open rectangular box has a volume of $V = 32$ cubic inches. What dimensions will make the surface area S a minimum? ***Note:*** $V = xyz = 32$; $S = xy + 2zy + 2zx$, where x and y are the dimensions of the rectangular base.

7. (a) Using the Lagrange multiplier method, find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

(b) Find the extreme values of f on the disk $x^2 + y^2 \leq 1$. ***Note:*** In view of problem 7(a), in problem 7(b) you only need additionally to find the extreme values of f on the interior of the disk $x^2 + y^2 \leq 1$, namely on $\{(x, y) \colon x^2 + y^2 < 1\}$.

8. The temperature $T(x, y, z)$ at the point $P(x, y, z)$ in three-space is $T = x^2 + 2y^2 + 3z^2$.

(a) Use differentials and approximate the temperature difference

$$T(3.1, 2.3, .98) - T(3, 2, 1)$$

between the points $(3.1, 2.3, .98)$ and $(3, 2, 1)$.

(b) Find a unit vector in the direction in which T increases most rapidly at the point $(3, 2, 1)$.

9. (a) Evaluate $\int_R \int f(x,y)\, dA = \int_1^2 \int_0^{x^2} (x+y)\, dy\, dx$.

(b) Sketch the region R in the x, y plane.

10. Given a solid which lies under the graph of the paraboloid $z = x^2 + y^2$, and over the rectangular region in the x, y plane having vertices $(0,0,0)$, $(0,1,0)$, $(2,0,0)$, $(2,1,0)$:

(a) Sketch a rough graph of the solid using a rectangular coordinate system.

(b) *Set up* an iterated double integral giving the volume of the solid.

(c) *Set up* an iterated triple integral giving the volume of the solid. Then evaluate the integral.

11. (a) *Set up* using polar coordinates (but do not integrate) an iterated double integral giving the area of the region which is inside the circle $r = 4\cos\theta$ and *outside* the circle $r = 2$.

(b) Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2+y^2)\, dy\, dx$ (where a is a positive constant) by changing to polar coordinates. ***Note:*** $y = \sqrt{a^2 - x^2}$ is the equation of a semicircle with center at $(0,0)$ and radius a.

12. (a) Let V denote the volume of the solid B bounded by the cone $z^2 = x^2 + y^2$ and the plane $z = 4$ (that is, B is the region above the cone and under the plane $z = 4$). Using cylindrical coordinates (r, θ, z), *set up* (no evaluation) a triple integral giving the volume of the solid B.

(b) Using spherical coordinates (ρ, θ, φ), set up and evaluate a triple integral to show that the volume of a sphere of radius a is $4/3\, \pi a^3$. ***Note:*** In spherical coordinates, an element of volume is $\nabla V = \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$.

ANSWERS TO SAMPLE EXAMINATION

1. (a) $\vec{r}(t) = (2\sin t)\vec{i} + 2(\cos t)\vec{j} + t^2\vec{k} = x\vec{i} + y\vec{j} + z\vec{k}$.

The velocity is $\vec{r}\,'(t) = (2\cos t)\vec{i} - 2(\sin t)\vec{j} + 2t\vec{k}$.

The speed $= |\vec{r}\,'(t)| = \sqrt{(2\cos t)^2 + (-2\sin t)^2 + (2t)^2} = \sqrt{4 + 4t^2} = 2\sqrt{1+t^2}$.

The point $P(t)$ never stops moving because the speed is never equal to 0 for any value of t.

(b) $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = \sin^2 t + \cos^2 t = 1 \Leftrightarrow x^2 + y^2 = 4$. This is (in three-space) the equation of a right circular cylinder with radius 2, and axis the z-axis. In cylindrical coordinates, the equation of the cylinder is $r = 2$.

2. (a) (i) $\rho = 5$ is the spherical coordinate equation of a sphere, with center at the origin and radius 5. Its rectangular, or Cartesian, coordinate equation is $x^2 + y^2 + z^2 = 25$.

(ii) It is probably easiest here to change $\rho = 2\cos\varphi$ first to Cartesian coordinates. Thus, multiply both sides of the preceding equation by ρ to get $\rho^2 = 2\rho\cos\theta$, from which $x^2 + y^2 + z^2 = 2z$. Now change this equation to cylindrical coordinates, getting $r^2 + z^2 = 2z$.

(iii) $r^2 + z^2 = 9$ changes to $x^2 + y^2 + z^2 = 9$, in Cartesian coordinates.

(b) (1) $x^2 + 5y^2 + z^2 = 25$; graph is an *ellipsoid.*

(2) $9x^2 - y^2 + 9z^2 = 9$; graph is an *hyperboloid of one sheet.*

(3) $\frac{x^2}{4} + \frac{y^2}{9} = z$; graph is an *elliptic paraboloid.*

3. (a) $$\frac{\partial f(x,y,z)}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h}.$$

(b) If $f(x,y,z) = (x + y^2 + z^3)^2$, then:

(i) $$\frac{\partial f(x,y,z)}{\partial x} = 2\,(x + y^2 + z^3)\cdot 1 = 2x + 2y^2 + 2z^3.$$

(ii) $$\frac{\partial^2 f(x,y,z)}{\partial y\,\partial x} = \frac{\partial}{\partial y}(2x + 2y^2 + 2z^3) = 4y.$$

(c) $f(x,y,z) = (x^3 + y^2 + z)^4 \Rightarrow fy\,(x,y,z) = 4\,(x^3 + y^2 + z)^3 \cdot 2y$

Therefore, $fy\,(0, 1, 1) = 4\,(0 + 1 + 1)^3\,(2) = 64$

4. $f(x,y) = x^2 + y^2$

(a) $\vec{\nabla} f(x,y) = \frac{\partial f}{dx}\vec{i} + \frac{\partial f}{dy}\vec{j} = 2x\vec{i} + 2y\vec{j}$, and $\vec{\nabla} f(-2,3) = -4\vec{i} + 6\vec{j}$

(b) A unit vector in the direction of $\vec{a} = 2\vec{i} - 3\vec{j}$ is $\vec{u} = \frac{\vec{a}}{|\vec{a}\,|} = \frac{2\vec{i} - 3\vec{j}}{\sqrt{4+9}} = \frac{2}{\sqrt{13}}\vec{i} - \frac{3}{\sqrt{13}}\vec{j}$.

Then the directional derivative of f at the point $(-2, 3)$ in the direction of the vector $\vec{a} = 2\vec{i} - 3\vec{j}$ is

$$D_{\vec{u}} f(-2,3) = \vec{\nabla} f(-2,3) \cdot \vec{u}$$
$$= (-4\vec{i} + 6\vec{j}) \cdot \left(\frac{2}{\sqrt{13}} \vec{i} - \frac{3}{\sqrt{13}} \vec{j} \right)$$
$$= -\frac{8}{\sqrt{13}} - \frac{18}{\sqrt{13}} = -\frac{26}{\sqrt{13}}$$

(c) The maximum rate of change of f at $(-2, 3)$ is $|\vec{\nabla} f(-2,3)| = |-4\vec{i} + 6\vec{j}| = \sqrt{(-4)^2 + 6^2} = \sqrt{52}$.

5. (a) For the surface with equation $z = x^2 + xy + y^2 = f(x,y)$, the equation of the tangent plane at the point $(1, 2, 7)$ is

$$z - 7 = \frac{\partial f(1,2,7)}{\partial x}(x-1) + \frac{\partial f(1,2,7)}{\partial y}(y-2).$$

Here $\frac{\partial f}{\partial x} = 2x + y$, $\frac{\partial f}{\partial y} = x + 2y$; so $\frac{\partial f}{\partial x}(1,2,7) = 4$, and $\frac{\partial f}{\partial y}(1,2,7) = 5$, and the equation of the tangent plane is $z - 7 = 4(x-1) + 5(y-2)$.

(b) Using the formula for a total derivative, we have

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = (2x+y)\cdot 2 + (x+2y)\cdot 3$$

at a general point (x, y, z). In particular, at the point $(1, 2, 7)$, $\frac{dz}{dt} = 4(2) + 5(3) = 23$.

6. (a) $u = x\sin y \Rightarrow u_x = \sin y,\ u_y = x\cos y$. Therefore $u_{xx} = 0,\ u_{xy} = \cos y,\ u_{yx} = \cos y,\ u_{yy} = -x\sin y$.

(b) First, write S as a function of x and y, noting $xyz = 32 \Rightarrow z = 32/xy$. Accordingly,

$$S = xy + 2y\left(\frac{32}{xy}\right) + 2x\left(\frac{32}{xy}\right)$$

or $$S = xy + \frac{64}{x} + \frac{64}{y}.$$

Next, find the critical points (x, y) by solving the system $\frac{\partial S}{\partial x} = 0,\ \frac{\partial S}{\partial y} = 0$, or here the equations $y - \frac{64}{x^2} = 0,\ x - \frac{64}{y^2} = 0.$

Observe $64 = yx^2 = xy^2 \Rightarrow xy(y-x) = 0 \Rightarrow x = y$. Hence $64 = x^3 = y^3$, from which $x = 4,\ y = 4$, and $z = 32/16 = 2$.

7. (a) We want to find the extreme value(s) of the function $f(x,y) = x^2 + 2y^2$, subject to the constraint $g(x,y) = x^2 + y^2 - 1 = 0$. Using the Lagrange multiplier method, solve the equations

$$\vec{\nabla} f(x,y) = \lambda \vec{\nabla} g(x,y), \text{ and } g(x,y) = 0 \text{ for } (x,y,\lambda).$$

We have, equivalently, to solve

(1) $\dfrac{\partial f}{\partial x} = \lambda \dfrac{\partial g}{\partial x} \Leftrightarrow 2x = 2\lambda x$

(2) $\dfrac{\partial f}{\partial y} = \lambda \dfrac{\partial g}{\partial y} \Rightarrow 4y = 2\lambda y$

(3) $x^2 + y^2 = 1$

Now from equation (1) above, we have $x = 0$, or $\lambda = 1$. If $x = 0$, then equation (3) gives $y = \pm 1$. If $\lambda = 1$, then equation (2) gives $y = 0$, and so, from equation (3), $x = \pm 1$. Therefore, there are four points on the circle $x^2 + y^2 = 1$ at which f has possible extreme values, namely the points $(0, 1)$, $(0, -1)$, $(1, 0)$, and $(-1, 0)$. Evaluating f at these four points, we see $f(0, 1) = 2$, $f(0, -1) = 2$, $f(1, 0) = 1$, and $f(-1, 0) = 1$. Hence, the maximum value of f on the circle $x^2 + y^2 = 1$ is $f(0, \pm 1) = 2$, and the minimum value is $f(\pm 1, 0) = 1$.

(b) Here, the system $\dfrac{\partial f}{\partial x} = 0$, $\dfrac{\partial f}{\partial y} = 0$, is $2x = 0$, $4y = 0$, and has the solution $x = 0$, $y = 0$. Hence, the only critical point for f on the interior of the disk is $(0, 0)$. Comparing the value $f(0, 0) = 0$ with the extreme values of f on the boundary of the disk, as found in 7(a) shows that the maximum value of f on the disk $x^2 + y^2 \le 1$ is $f(0, \pm 1) = 2$ and the minimum value is $f(0, 0) = 0$.

8. (a) Given the temperature $T(x,y,z)$ at a point $P(x,y,z)$ is $T(x,y,z) = x^2 + 2y^2 + 3z^2$, the differential dT of T at the point P, corresponding to increments dx, dy, dz in the values of x, y, z, respectively, is

$$\begin{aligned} dT &= \frac{\partial T(x,y,z)}{\partial x} dx + \frac{\partial T(x,y,z)}{\partial y} dy + \frac{\partial T(x,y,z)}{\partial z} dz \\ &= 2x\,dx + 4y\,dy + 6z\,dz. \end{aligned}$$

We want to approximate the exact change ∇T in temperature as (x,y,z) changes from $(3, 2, 1)$ to $(3.1, 2.3, 0.98)$, by dT. Taking $x = 3$, $y = 2$, $z = 1$, $dx = 0.1$, $dy = 0.3$, $dz = -0.02$, we get the desired approximation,

$$dT = 2\,(3)\,(0.1) + 4\,(2)\,(0.3) + 6\,(1)\,(-0.02) = 2.88$$

(b) The temperature $T(x, y, z)$ increases most rapidly at the point $(3, 2, 1)$ in the direction of the gradient $\vec{\nabla}\,(3, 2, 1)$. Now the gradient $\vec{\nabla}T(x, y, z)$ of T at a general point (x, y, z) is $\vec{\nabla}T(x, y, z) = 2x\vec{i} + 4y\vec{j} + 6z\vec{k}$. Taking $x = 3,\ y = 2,\ z = 1$, we see $\vec{\nabla}\,T(3, 2, 1) = 6\vec{i} + 8\vec{j} + 6\vec{k}$. Since then $|\vec{\nabla}\,T(3, 2, 1)| = \sqrt{6^2 + 8^2 + 6^2} = \sqrt{136}$, the desired unit vector is therefore $\dfrac{\vec{\nabla}\,T(3, 2, 1)}{|\vec{\nabla}\,T(3, 2, 1)|} = \dfrac{6}{\sqrt{136}}\vec{i} + \dfrac{8}{\sqrt{136}}\vec{j} + \dfrac{6}{\sqrt{136}}\vec{k}$.

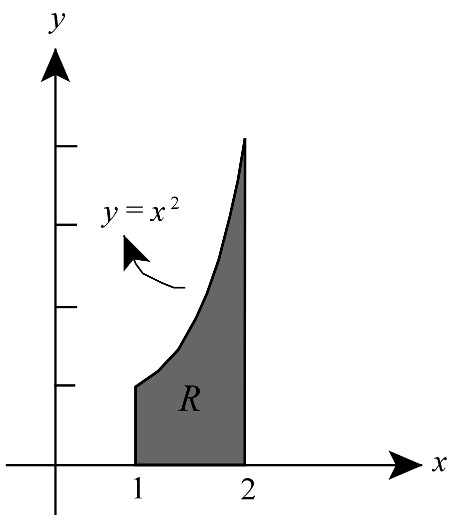
9. (a)

$$\int_R\!\!\int f(x, y)\,dA = \int_1^2\int_0^{x^2}(x + y)\,dy\,dx = \int_1^2\left[xy + \frac{y^2}{2}\right|_{y=0}^{y=x^2}dx$$

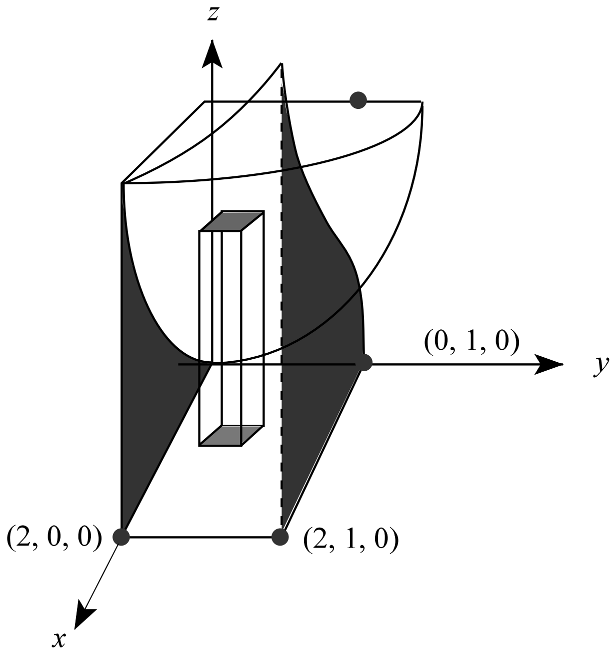
$$= \int_1^2\left(x^3 + \frac{x^4}{2}\right)dx = \frac{x^4}{4} + \frac{x^5}{10}\Big|_1^2$$

$$= \frac{16}{4} + \frac{32}{10} - \left(\frac{1}{4} + \frac{1}{10}\right) = \frac{31}{10} + \frac{15}{4} = \frac{137}{20}$$

(b)



10. (a) This solid is rather difficult to picture. All you need to do is to get a good enough picture in your mind to set up the proper limits of integration.



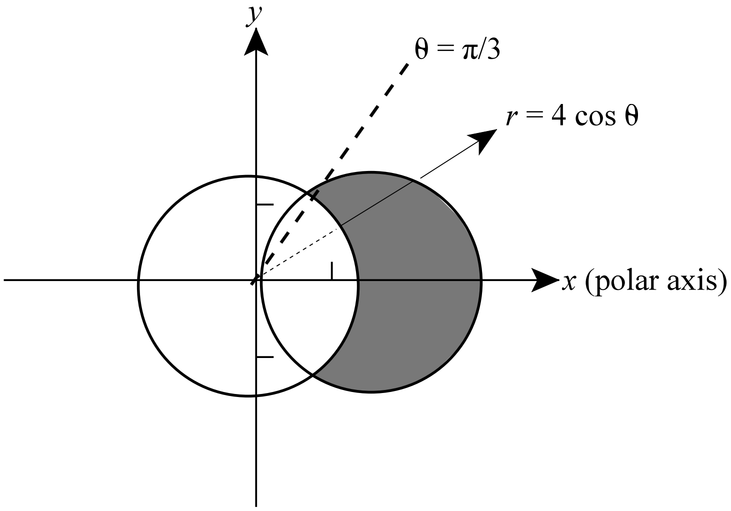
(b) Volume $= \int_0^2 \int_0^1 (x^2 + y^2)\, dy\, dx$

(c) Volume $= \int_0^2 \int_0^1 \int_0^{(x^2+y^2)} dz\, dy\, dx = \int_0^2 \int_0^1 \left[z \Big|_{z=0}^{z=x^2+y^2} \right] dy\, dx$

$$= \int_0^2 \int_0^1 (x^2 + y^2)\, dy\, dx = \int_0^2 \left[x^2 y + \frac{y^3}{3} \Big|_{y=0}^{y=1} \right] dx$$

$$= \int_0^2 \left(x^2 + \frac{1}{3} \right) dx = \frac{x^3}{3} + \frac{x}{3} \Big]_0^2 = \frac{10}{3}.$$

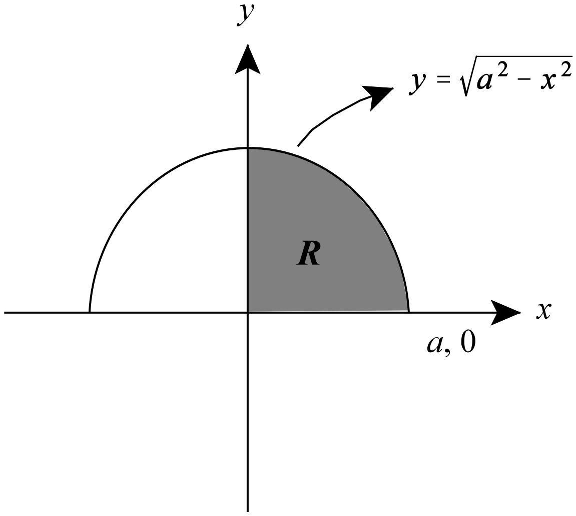
11. (a) First, draw a rough graph of the circles $r = 2$ and $r = 4 \cos \theta$



The circles intersect where $2 = 4\cos\theta \Leftrightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$. The area inside circle $r = 4\cos\theta$ and outside circle $r = 2$ is

$$2 \cdot \int_0^{\pi/3} \int_2^{4\cos\theta} r\,dr\,d\theta = 2 \text{ (area of shaded region in the first quadrant).}$$

11. (b) From the limits of integration on the given iterated double integral, $I = \int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2+y^2)\,dy\,dx$, we see that I is equal to the double integral $\int_R\int (x^2+y^2)\,dA$ of x^2+y^2 over a region *R* in the x, y plane. *R* is bounded by the graphs of $y = \sqrt{a^2-x^2}$, a semicircle with center at $(0, 0)$ and radius *a*, and the lines $x = 0$ and $y = 0$, as depicted below.



Changing to polar coordinates, the equation of the semicircle is $r = a$, and the lines $x = 0,\ y = 0$ become $\theta = \pi/2,\ \theta = 0$, respectively.

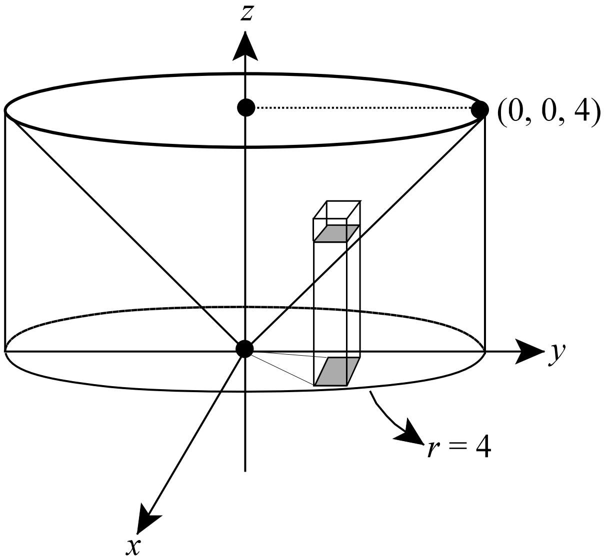
Then, replacing x^2+y^2 by $=r^2$ in the integrand of integral *I*, and $dy\,dx$ by $r\,dr\,d\theta$, we find

$$I = \int_0^{\pi/2} \int_0^a r^2\, r\,dr\,d\theta = \int_0^{\pi/2} \frac{r^4}{4}\Big]_0^a d\theta$$

$$= \frac{a^4}{4}\int_0^{\pi/2} d\theta = \frac{a^4}{4}\,\theta\Big]_0^{\pi/2}$$

$$= \frac{a^4}{4}\cdot\frac{\pi}{2} = \frac{\pi a^4}{8}$$

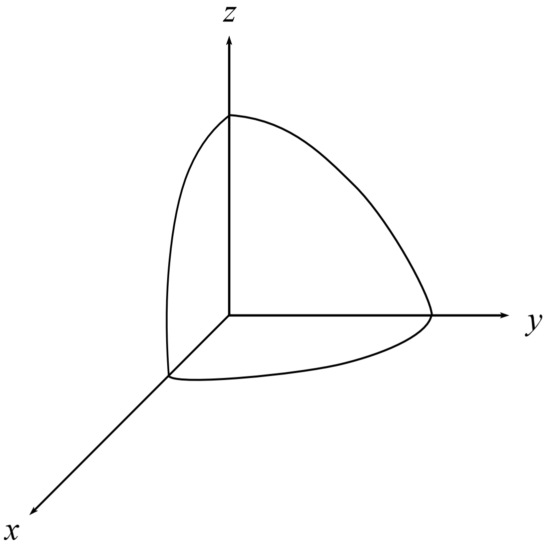
12. (a) It is helpful to first draw a rough graph of the solid *B*. Observe the cone $z^2 = x^2 + y^2$ has circular cross-sections cut out by planes parallel to the x, y plane. In particular, the cross-section cut out by the plane $z = 4$ is the circle $x^2 + y^2 = 16$, of radius 4.



In cylindrical coordinates, the equation of the cone is $z^2 = r^2$, or just $z = r$ here. The volume of the solid *B* is Vol $= \int_0^{2\pi} \int_0^4 \int_r^4 dz\, r\, dr\, d\theta$.

Note that after the first integration on *z*, the remaining integrations on *r* and θ are over the disk $r \leq 4$ in the x, y plane.

(b)



In spherical coordinates, the equation of a sphere with center at the origin and radius a is $\rho = a$. The volume is 8 (volume of sphere in first octant), or

$$
\begin{aligned}
V &= 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta \\
&= 8\int_0^{\pi/2}\int_0^{\pi/2} \frac{\rho^3}{3}\bigg|_0^{a} \sin\varphi \, d\varphi \, d\theta = \frac{8a^3}{3}\int_0^{\pi/2} -\cos\varphi\bigg|_0^{\pi/2} d\theta \\
&= \frac{8a^3}{3}\int_0^{\pi/2}\left[-\cos\frac{\pi}{2} + \cos 0\right] d\theta \\
&= \frac{8a^3}{3}\int_0^{\pi/2}[0+1]\, d\theta = \frac{8a^3}{3}\,\theta\bigg]_0^{\pi/2} = \frac{4}{3}\pi a^3 .
\end{aligned}
$$