



MATH 263ACCE — SAMPLE EXAMINATION

Take the self-check sample examination which follows and check your answers using the key that follows the exam.

You will gain the most benefit from this sample exam if you take it as if it were the supervised examination. Set yourself a time limit of 3 hours and do not use your textbook or notes. Practice writing out all the steps in your solutions, since you will have to show your work on the problems for the supervised examination.

1. Find the equation of the tangent line to the parabola with equation $y = x^2$, if the tangent line is parallel to the line with equation $y = 4x + 2$.

2. Evaluate $\lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$ in two ways:

- (1) directly, using algebra and limit laws;
- (2) using the definition of a derivative as a limit.

3. (a) Is the function f defined by

$$f(x) = \begin{cases} \frac{x}{|x|} & \text{for } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \text{ continuous at } x = 0?$$

Explain briefly.

- (b) For the function f defined as

$$f(x) = \frac{2x \cos x - x^2}{2x}, \text{ if } x \neq 0$$

how could you define f at $x = 0$ to make f continuous at $x = 0$?

4. If a ball is thrown vertically upward with a velocity of 80 feet per second, then its height after t seconds is $s = 80t - 16t^2$.
 - (a) what is the maximum height reached by the ball?
 - (b) what is the velocity of the ball when it is 96 feet above the ground on its way up? on its way down?

5. Compute the derivative $D_x y$ for:

(a) $y = \frac{x+1}{x-1}$ (b) $y = \left(\frac{x+1}{x-1}\right)^{10}$

6. Find the derivative $D_x y$ if $y = (\sin x - \cos x)^2$.

7. Using implicit differentiation, compute $D_x y = y^1$ if $x^2 y + y^2 = x$.

8. Differentiate $f(x) = \left(3x + \sqrt[3]{1-x^2}\right)^4$.

9. (a) Beginning with the identity $\tan x = \frac{\sin x}{\cos x}$, derive the formula $D_x \tan x = \sec^2 x$.

(b) Find the derivative $D_x y$ for $y = x \tan(x^2)$.

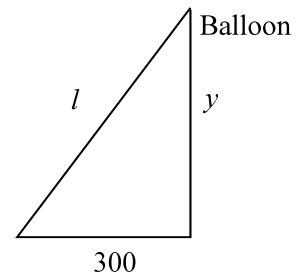
10. (a) Use differentials to approximate $\sqrt[3]{27.5}$. *Hint:* First, let $y = \sqrt[3]{x}$.

(b) Explain, using the geometrical meaning of differentials, whether the approximation of $\sqrt[3]{27.5}$ obtained using differentials is larger or smaller than the exact value of $\sqrt[3]{27.5}$.

11. (a) A point $P(x, y)$ moves on the graph of $y^2 - x^2 = 9$ such that $\frac{dx}{dt} = \frac{1}{x}$ (here t is times).

Find $\frac{dy}{dt}$ at the point $(4, 5)$.

(b) A weather balloon is rising vertically at a rate of 2 ft./second. An observer is situated 300 feet from a point on the ground directly below the balloon. At what rate is the distance l between the balloon and the observer changing when the altitude y of the balloon is 500 feet?



12. (a) For the function f defined as $f(x) = \frac{1}{2}x - \sin x$, find the local extrema of f on the interval $(0, 2\pi)$ using the Second Derivative Test.

(b) Find the point(s) of inflection of the graph of f on the interval $(0, 2\pi)$.

13. (a) Find the critical numbers of the function f defined as $f(x) = 4x^3 - 3x^4$. Using the First Derivative Test, determine whether f has local maximum or local minimum values, or neither, at these critical numbers. *Explain.*

(b) What is the global (absolute) minimum and global (absolute) maximum values of f on the interval $[-2, 2]$?

14. (a) For the function f , with $f(x) = \frac{5x}{4 - x^2}$, compute

$$\lim_{x \rightarrow \pm\infty} f(x), \quad \lim_{x \rightarrow 2^+} f(x), \quad \lim_{x \rightarrow -2^+} f(x)$$

- (b) What are the equations of the horizontal and vertical asymptotes to the graph of f ?

15. The first and second derivatives of $f(x) = \frac{5x}{4 - x^2}$ (see Problem 14 (a)) above are:

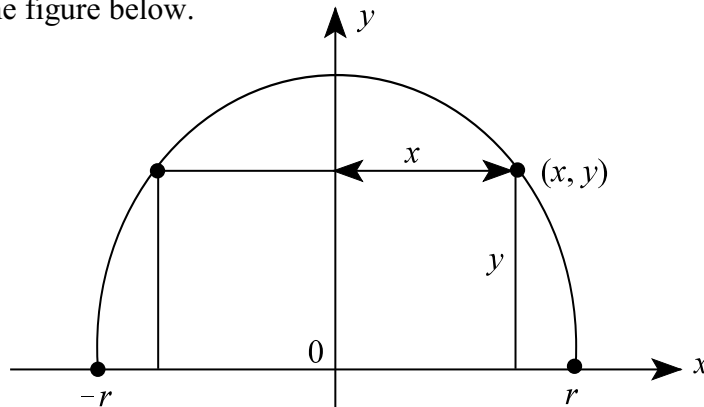
$$f'(x) = \frac{5(x^2 + 4)}{(4 - x^2)^2}, \quad f''(x) = \frac{10x(x^2 + 12)}{(2 - x)^3(2 + x)^3}$$

- (a) Analyze the graph of the function f as to:

(1) domain of f ; (2) symmetry; (3) intercepts; (4) where increasing or decreasing; (5) intervals on which the graph is concave up or concave down.

- (b) Sketch the graph of the function f .

16. Consider a semicircle of radius r which is the upper half of the circle $x^2 + y^2 = r^2$, with center the origin, as in the figure below.



A rectangle has two of its vertices on the semicircle and the other two on the x -axis, as in the figure above. Then the rectangle has sides of lengths $2x$ and y , so its area is $A = 2xy$, with $0 \leq x \leq r$. Of all such rectangles, find the dimension of the rectangle of maximum area.

17. (a) For the three functions f defined on the indicated closed intervals, state whether the *Mean Value Theorem* applies. If it does not apply, state a reason why.

(1) $f(x) = |x|$, $[-1, 1]$

(2) $f(x) = \sqrt{x}$, $[0, 1]$

(3) $f(x) = x^{1/3}$, $[-1, 1]$

- (b) For $f(x) = \sin x$, $-\pi \leq x \leq \pi$, find all numbers c in the open interval $(-\pi, \pi)$ satisfying the conclusion of the Mean Value Theorem. Then sketch a graph of $y = \sin x$, on $[-\pi, \pi]$, and interpret your answer geometrically.

ANSWER KEY FOR SAMPLE EXAMINATION

1. The slope of the given line is 4. The slope of the tangent line at a point (x,y) on the parabola is $D_x y = 2x$. Thus, the given line and the tangent will be parallel if and only if $2x = 4$, or $x = 2$, and $y = 2^2 = 4$. By the point-slope equation of a line, the equation of the tangent line is $y - 4 = 4(x - 2)$.

2. (1) In the given limit, first observe that the numerator and denominator of the fraction $(\sqrt{x+h} - \sqrt{x})/h$ both approach zero as $h \rightarrow 0$. To avoid this so-called *indeterminate form* of the form $0/0$ at $h = 0$, you need to algebraically change the form of the fraction $(\sqrt{x+h} - \sqrt{x})/h$. The “trick” is to multiply the numerator and denominator by $(\sqrt{x+h} + \sqrt{x})$, and then simplify. We get

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} &= \lim_{h \rightarrow 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \left(\frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} \right) = \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}. \end{aligned}$$

(2) From the definition of a derivative as a limit,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h},$$

it is clear that the given limit is the derivative of $f(x) = \sqrt{x} = x^{1/2}$, namely

$$f'(x) = \frac{1}{2\sqrt{x}}, \text{ for } x > 0$$

3. (a) It is convenient here to note that $f(x) = \frac{x}{|x|}$ can be written as

$$f(x) = \begin{cases} \frac{x}{x} = 1, & \text{if } x > 0 \\ \frac{x}{-x} = -1, & \text{if } x < 0 \end{cases}$$

Hence, $\lim_{x \rightarrow 0^+} f(x) = 1 \neq \lim_{x \rightarrow 0^-} f(x) = -1$. Since the right-hand and left-hand limits of f at $x = 0$ are not equal, $\lim_{x \rightarrow 0} f(x)$ does not exist. Therefore, f is not continuous at $x = 0$ (even though f is defined at $x = 0$).

$$(b) \quad \lim_{x \rightarrow 0} \frac{2x \cos x - x^2}{2x} = \lim_{x \rightarrow 0} \frac{2 \cos x - x}{2} = \frac{2(1) - 0}{2} = 1$$

Therefore, f will be continuous at 0 if we define $f(0) = 1$.

4. (a) Since $s(t) = 80t - 16t^2$, the velocity of V of the ball at time t is $V(t) = \frac{ds}{dt} = 80 - 32t$.

Now the maximum height will be reached when

$$V(t) = 0 \iff 80 - 32t = 0 \implies t = 2.5 \text{ seconds. Thus, the maximum height is}$$

$$s(2.5) = 80(2.5) - 16(2.5)^2 = 100 \text{ feet.}$$

(b) First, set $s = 96$, and solve the equation

$$96 = 80t - 16t^2 \iff 16(t^2 - 5t + 6) = 0 \iff (t - 2)(t - 3) = 0 \implies t = 2, t = 3;$$

so the ball is on its way up at a height of 96 feet when $t = 2$, and on its way down at a height of 96 feet when $t = 3$. The velocity at $t = 2$ seconds and $t = 3$ seconds is, respectively,

$$V(2) = 80 - 32(2) = 16 \text{ ft./sec., and}$$

$$V(3) = 80 - 32(3) = -16 \text{ ft./sec.}$$

Remark: After the ball has reached its maximum height at $t = 2.5$ seconds, that is, for $t > 2.5$, the velocity $V = \frac{ds}{dt}$ is negative because s is a decreasing function of t .

5. (a)
$$D_x y = D_x \left(\frac{x+1}{x-1} \right) = \frac{(x-1)D_x(x+1) - (x+1)D_x(x-1)}{(x-1)^2}$$

$$= \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

(b)
$$D_x \left(\frac{x+1}{x-1} \right)^{10} = 10 \left(\frac{x+1}{x-1} \right)^9 \cdot D_x \left(\frac{x+1}{x-1} \right)$$
. Now use the result in 5(a) to get

$$D_x y = 10 \left(\frac{x+1}{x-1} \right)^9 \left[\frac{-2}{(x-1)^2} \right].$$

6. Applying first the *Power Rule* for functions, we have

$$D_x y = D_x (\sin x - \cos x)^2 = 2(\sin x - \cos x) \cdot D_x (\sin x - \cos x)$$

$$= 2(\sin x - \cos x)(\cos x + \sin x)$$

$$= 2(\sin^2 x - \cos^2 x)$$

Remarks: An alternative way of computing $D_x y$ is to do a little algebra first, and use a trigonometric identity, as follows:

$$y = (\sin x - \cos x)^2 = \sin^2 x + \cos^2 x - 2 \sin x \cos x \\ = 1 - 2 \sin x \cos x$$

Then,

$$D_x y = D_x (1 - 2 \sin x \cos x) = -2 D_x (\sin x \cos x) \\ = -2 [(\sin x) D_x \cos x + (\cos x) D_x \sin x] \\ = -2 [(\sin x)(-\sin x) + (\cos x)(\cos x)] \\ = -2 (\sin^2 x - \cos^2 x)$$

7. Assuming that the given equation $x^2 y + y^2 = x$ defines y as a differentiable function of x , take the derivative of both sides of the equation with respect to x , and equate the results. We obtain

$$D_x (x^2 y + y^2) = D_x x \\ x^2 \cdot D_x y + y \cdot D_x x^2 + D_x (y^2) = 1 \\ x^2 \cdot D_x y + 2xy + 2y \cdot D_x y = 1$$

Now solve the preceding equation for $D_x y$, getting

$$D_x y (x^2 + 2y) = 1 - 2xy \\ D_x y = \frac{1 - 2xy}{x^2 + 2y}$$

8. $f(x) = \left(3x + \sqrt[3]{1 - x^2}\right)^4 = [3x + (1 - x^2)^{1/3}]^4$

$$\Rightarrow f'(x) = 4 [3x + (1 - x^2)^{1/3}]^3 \cdot D_x [3x + (1 - x^2)^{1/3}] \\ = 4 \left(3x + \sqrt[3]{1 - x^2}\right)^3 \cdot \left[3 + \frac{1}{3} (1 - x^2)^{-2/3} (-2x)\right]$$

9. (a) $D_x \tan x = D_x \left(\frac{\sin x}{\cos x}\right) = \frac{(\cos x) D_x \sin x - (\sin x) D_x \cos x}{\cos^2 x}$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{(\cos x)^2} = \left(\frac{1}{\cos x}\right)^2 \\ = \sec^2 x$$

$$\begin{aligned}
 \text{(b) } y = x \tan x^2 &\implies D_x y = D_x (x \tan x^2) \\
 &= x \cdot D_x \tan(x^2) + [\tan(x^2)] \cdot D_x x \\
 &= [x \sec^2(x^2)] \cdot 2x + \tan(x^2)
 \end{aligned}$$

Note: In computing $D_x \tan(x^2)$ as outlined above, we have mentally applied the Chain Rule. To be more explicit, you could let $u = x^2$, and by the Chain Rule, write

$$\begin{aligned}
 D_x \tan(x^2) &= (D_u \tan u) \cdot D_x u \\
 &= (\sec^2 u) D_x x^2 = 2x \sec^2(x^2).
 \end{aligned}$$

10. (a) Recall for a general function f ,

$$\Delta y = f(x + \Delta x) - f(x) \iff f(x + \Delta x) = f(x) + \Delta y$$

Now for Δx small, $\Delta y \approx dy$; so that

$$\begin{aligned}
 f(x + \Delta x) &= f(x) + \Delta y \approx f(x) + dy \\
 &= f(x) + f'(x) \cdot \Delta x
 \end{aligned}$$

In particular, for $y = f(x) = \sqrt[3]{x} = x^{1/3}$, we have

$$\begin{aligned}
 f'(x) &= \frac{1}{3} x^{-2/3} = \frac{1}{3 \left(\sqrt[3]{x} \right)^2}, \text{ and} \\
 \sqrt[3]{x + \Delta x} &\approx \sqrt[3]{x} + \frac{1}{\left(3 \sqrt[3]{x} \right)^2} \cdot \Delta x
 \end{aligned}$$

At this point, it is most appropriate to take $x = 27$, $\Delta x = 0.5$ (because 27 is the number closest to 27.5 for which we find the cube root without using a calculator—which is the idea here). This yields

$$\begin{aligned}
 \sqrt[3]{27.5} &\approx \sqrt[3]{27} + \frac{1}{\left(3 \sqrt[3]{27} \right)^2} (0.5) \\
 &= 3 + \frac{1}{3(9)} (0.5) = 3 + \frac{1}{54} \approx 3.0185
 \end{aligned}$$

- (b) The answer here hinges on the concavity of the graph of $y = \sqrt[3]{x}$ in an open interval I containing $x = 27$. Now for $f(x) = x^{1/3}$, we have $f''(x) = -\frac{2}{9}x^{-5/3} < 0$ when the interval I contains only positive numbers. Hence the graph of f is concave down on interval I. Therefore, the tangent line to the graph of $y = \sqrt[3]{x}$ at the point $(27, 3)$ is above the graph. Accordingly, the approximation 3.0185 for $\sqrt[3]{27.5}$ is too large. Incidentally, using a calculator yields $\sqrt[3]{27.5} = 3.0184$.

11. (a) Begin by differentiating both sides of $y^2 - x^2 = 9$ with respect to t . This gives

$$\begin{aligned} \frac{d}{dt}(y^2 - x^2) &= \frac{d}{dt}(9) \iff 2y \cdot \frac{dy}{dt} - 2x \cdot \frac{dx}{dt} = 0 \\ &\iff y \frac{dy}{dt} - x \left(\frac{1}{x} \right) = 0 \end{aligned}$$

Set $x = 4$, $y = 5$, getting $5 \frac{dy}{dt} = 1$, and $\frac{dy}{dt} = \frac{1}{5}$.

- (b) You are given $\frac{dy}{dt} = 2$, for $t > 0$, and want to find $\frac{dL}{dt}$ when $y = 500$. From the

Pythagorean theorem, $l = \sqrt{y^2 + (300)^2} = [y^2 + (300)^2]^{1/2}$. Thus,

$$\frac{dL}{dt} = \frac{1}{2}[y^2 + (300)^2]^{-1/2} \cdot 2y \frac{dy}{dt} = \frac{2y}{\sqrt{y^2 + (300)^2}}. \text{ In particular, when } y = 500,$$

$$\frac{dL}{dt} = \frac{2(500)}{\sqrt{(500)^2 + (300)^2}} = \frac{1000}{100\sqrt{34}} = \frac{10}{\sqrt{34}} \text{ ft./second.}$$

12. (a) $f(x) = \frac{1}{2}x - \sin x \implies f'(x) = \frac{1}{2} - \cos x$, $f''(x) = \sin x$. The critical points of f are the values of x in the interval $(0, 2\pi)$ satisfying $f'(x) = 0 \iff \cos x = \frac{1}{2}$, namely

$$x = \frac{\pi}{3}, \text{ and } x = \frac{5\pi}{3}.$$

Here $f''(x)\left(\frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, and $f''\left(\frac{5\pi}{3}\right) = \sin\frac{5\pi}{3} = \frac{-\sqrt{3}}{2}$. Hence, by the second derivative test, f has a local minimum at $x = \frac{\pi}{3}$ and a local maximum at $x = \frac{5\pi}{3}$, namely

$$f\left(\frac{\pi}{3}\right) = \frac{1}{2} \cdot \frac{\pi}{3} - \frac{\sqrt{3}}{2}, \text{ and } f\left(\frac{5\pi}{3}\right) = \frac{1}{2} \cdot \frac{5\pi}{3} + \frac{\sqrt{3}}{2}.$$

(b) $f''(x) = 0 \iff \sin x = 0 \implies x = \pi$. Also, the second derivative changes sign as x increases through π (and so the concavity of f changes at $x = \pi$). Thus, the point $\left(\pi, \frac{1}{2}\pi\right)$ is an inflection point.

13. (a) $f(x) = 4x^3 - 3x^4 \implies f'(x) = 12x^2 - 12x^3 = 12x^2(1 - x)$.

$f'(x) = 0$ at $x = 0, x = 1$, which are the critical numbers for f .

For $x = 0$, $f'(x)$ doesn't change sign as x increases through 0; hence there is no local extremum of f at $x = 0$.

For $x = 1$, $f'(x)$ changes sign from positive to negative as x increases through 1. Thus, $f(1) = 1$ is a local maximum value of f .

(b) Just note $f(-2) = -80, f(2) = -16, f(1) = 1$. Comparing these values, we see that the global (or absolute) maximum value of f on $[-2, 2]$ is 1, and the absolute minimum value of f on $[-2, 2]$ is -80.

14. (a) For $f(x) = \frac{5x}{4 - x^2} = \frac{5x}{(2 - x)(2 + x)}$, $\lim_{x \rightarrow \pm\infty} f(x) = 0$. To compute $\lim_{x \rightarrow 2^+} f(x)$, observe that x

is just a little larger than 2, the factors $5x$ and $2 + x$ are positive, and the factor $(2 - x)$ is close to zero, but negative. Hence, $\lim_{x \rightarrow 2^+} f(x) = -\infty$.

Similarly, to compute $\lim_{x \rightarrow -2^+} f(x)$, first take a value of x close to -2, but greater than -2,

such as $x = -1.9$. For such a value of x , the factor $5x$ is negative, the factor $(2 + x)$ is positive but close to zero, and the factor $(2 - x)$ is positive. This kind of analysis indicates that $\lim_{x \rightarrow -2^+} f(x) = -\infty$.

- (b) The equation of the horizontal asymptote is $y = 0$. There are two vertical asymptotes, one with equation $x = 2$, and one with equation $x = -2$.

15. (a) (1) domain of f = set of all real numbers x except $x = \pm 2$; that is, domain of $f = (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

(2) In the equation $y = \frac{5x}{4 - x^2}$, if we replace x by $-x$, the equation changes.

From this, it follows that the graph is not symmetric with respect to the y -axis. Also, if we replace x by $-x$, and y by $-y$, the equation doesn't change. Therefore, the graph is symmetric with respect to the origin.

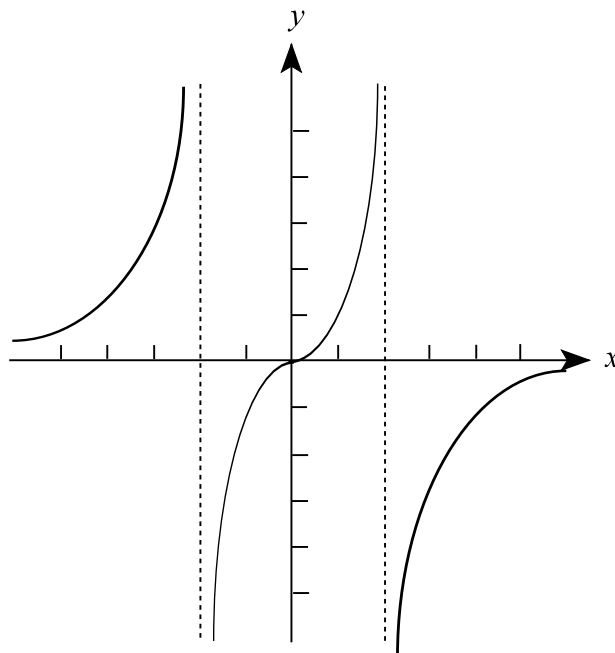
(3) Intercepts are $x = 0$, and $y = 0$ (so the graph passes through the origin).

(4) Since $f'(x) > 0$ for all x (except for $x = \pm 2$, where f and f' are not defined), f is increasing for all x .

(5) Observe $f''(x) < 0$ for $x > 2$ and for $-2 < x < 0$:
 $f''(x) > 0$ for $0 < x < 2$, and for $x < -2$.

It follows from the signs of $f''(x)$ that the graph of f is concave down on $(2, \infty) \cup (-2, 0)$ and concave up for $(0, 2) \cup (-\infty, -2)$.

(b)



16. The *essential* first step here is to write $A = 2xy$ as a function of one variable, say x , by writing y as a function of x . From the equation $x^2 + y^2 = r^2$, we have $y^2 = r^2 - x^2$, with $y \geq 0$. Thus, $y = \sqrt{r^2 - x^2}$ (the equation of the bottom half of the circle, which we are not interested in here, is $y = -\sqrt{r^2 - x^2}$). Hence, the area of a typical inscribed rectangle in the semicircle is

$$A(x) = 2x\sqrt{r^2 - x^2} = 2x(r^2 - x^2)^{1/2}, \quad 0 \leq x \leq r$$

It follows that

$$\begin{aligned} A'(x) &= 2x \cdot \frac{1}{2} (r^2 - x^2)^{-1/2} (-2x) + 2 (r^2 - x^2)^{1/2} \\ &= \frac{-2x^2}{\sqrt{r^2 - x^2}} + 2\sqrt{r^2 - x^2} \\ &= \frac{-2x^2 + 2(r^2 - x^2)}{\sqrt{r^2 - x^2}} = \frac{2r^2 - 4x^2}{\sqrt{r^2 - x^2}} \end{aligned}$$

Solving $A'(x) = 0$ yields $x^2 = r^2/2$, or $x = r/\sqrt{2}$. Clearly this critical value of x maximizes $A(x)$ on $[0, r]$ because $A(0) = A(r) = 0$, and $A(r/\sqrt{2})$ is positive. The rectangle of maximum area thus has dimensions $2 \cdot r/\sqrt{2} = \sqrt{2}$ by $r/\sqrt{2}$.

17. (a) (1) The Mean Value theorem does not apply to $f(x) = |x|$ on the interval $[-1, 1]$ because f is not differentiable at $x = 0$.
- (2) For $f(x) = \sqrt{x}$, we have $f'(x) = \frac{1}{2\sqrt{x}}$. The Mean Value theorem does apply to f on interval $[0, 1]$.
- (3) $f(x) = x^{1/3}$, then $f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$. The Mean Value theorem does not apply here because $f'(0)$ doesn't exist.
- (b) First of all, $f(x) = \sin x$ is continuous on $[-\pi, \pi]$, and $f'(x) = \cos x$ exists on $(-\pi, \pi)$. Therefore, the Mean Value theorem applies. Accordingly, the conclusion of the Mean Value theorem states there is at least a number c in $(-\pi, \pi)$ such that

$$f'(c) = \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)} \iff \cos c = \frac{0 - 0}{2\pi} = 0$$

It follows that $c = \pm\pi/2$.

