MATH 163A CCE — SAMPLE EXAMINATION

Directions: Allow yourself three hours to work the problems. You will gain the most benefit from this test if you do not use your textbook or notes. You may, however, use your calculator. Work the problems neatly and carefully and use graph paper where needed.

Work all the problems, then check them with the answer key that follows the test. Total your points and carefully review any weak points before you apply for the supervised examination.

1. (5 Points) Answer the following with "TRUE or "FALSE."

(a) If $f(x) = x^3$, then $f(x+h) = x^3 + h^3$

(b) The lines: $2x - 3y + 12 = 0$ and $3x + 2y + 6 = 0$ are perpendicular to each other.

(c) The domain of the function $f(x) = \sqrt{x+4}$ is all real numbers.

(d) The instantaneous rate of change of f at $x = a$ is given by

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

(e) The partial derivative $f_x \equiv \frac{\delta y}{\delta x}$ of a function $f(x, y)$ gives the rate of change of the function with respect to the independent variable x.

2. (10 Points) In the blanks write the word or formula (in capital letters) to make the following statements correct.

(a) If $f(x) = 3x^{-3} + \sqrt{x} - \frac{1}{\sqrt{x}} + \frac{5}{x}$ then $f'(x) =$ __________________________.

(b) If $f(x, y) = x^3y^3 + \frac{x}{y} - \frac{y}{x}$, then $f_x(2, 1) =$ __________________________.

(c) The $\lim_{x \to \infty} \frac{4x}{2x - 7}$ is equal to __________________________.

(d) If the profit function is $P(x) = 3 + 100x - x^2$, then marginal profit at $x = 10$ is ______________________.

(e) For the sphere: $(x-1)^2 + (y+2)^2 + (z-3)^2 = 144$, the radius is ____________ and center is located at the point C (____________) .

3. (10 Points) Find the derivatives of the following functions. Do not simplify the answers.

(a) $f(x) = \left(x^2 + \frac{1}{x^2}\right)\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$

(b) $g(x) = \left(\frac{x^2 + 3x}{2x - 3}\right)^{1/3}$

(c) $h(x) = 8(20x^2 - 4x + 10)^{-8}$

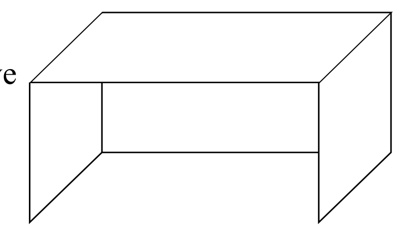
4. (10 Points) Let $f(x,y) = 6x^4 + 5y^2 - 3x^2y^3$. Find the following partial derivatives:

(a) f_{xx}

(b) f_{yy}

(c) $f_{xy}(-1,-2)$

5. (10 Points) A housing complex wants to build a school bus stop shelter for the children. The shelter is designed to have square side walls, a rectangular back wall, and a flat roof (see figure). Find the dimensions with a capacity of 1024 cubic feet that will require the minimum amount of materials.



6. (10 Points) An apple grower finds that average yield is 60 bushels per tree when 20 are planted in the orchard. Each additional tree (up to 15) decreases the average yield per tree by 2 bushels. How many trees will maximize the total yield of the orchard? With this many trees, what will be the average yield per tree, and the total yield of the orchard?

7. (10 Points) Fido Packaging Company wants to supply to a customer a box that is closed on the top and has a capacity of 1400 cubic feet. The cost of making the top is \$8.00 per square foot; the cost of making the bottom is \$6.00 per square foot, the cost is \$5.00 per square foot to make all four sides. Find the dimension of the box which will have the minimum cost. (Hint: volume = *xyz*; use partial derivatives f_x, f_y, etc.)

8. (10 Points) Using the method of Lagrange Multipliers, solve the following optimization problem: A manufacturer makes three different products: A, B, and C, on which the profit per unit is \$6, \$8, and \$10, respectively. The research department has decided that *x, y,* and *z* must be subject to the constraint: $x^2 + y^2 + z^2 = 800$. Find the levels of production of each product that will maximize daily profit.

9. (10 Points)

(a) Find the equation of a straight line which is perpendicular to the line $3x - y + 5 = 0$ and passes through the point $P(-1, 4)$.

(b) Using calculus, find the slope of the tangent line to the curve $y = 4 - x^2$ at the point $P(1,3)$. Next, write the equation of this tangent line.

10. (10 Points) The number of bacteria in a culture is given the function $B(t) = 500 + 8t^3$ where t is measured in days.

(a) How many bacteria would there be after 3 days and how many were there to begin with?

(b) In how many days would the bacteria be tripled?

(c) At what rate is the bacteria increasing?

11. (10 Points) Using the curve-sketching techniques of calculus, sketch the graph of the function $P(x) = (x^2 - 4)^2$, on the interval [0,3]. Be sure to give details of: x-intercept, y-intercepts, symmetry, critical points, increasing/decreasing intervals, points of inflection, concave up/concave down intervals, local extreme values, and absolute extreme values.

12. (10 Points) The cost in dollars of producing x number of items is given by the cost function $C(x) = 2x + 56$, and its demand function at price p is $p = 20 - x$.

(a) Find the average cost of producing eight items.

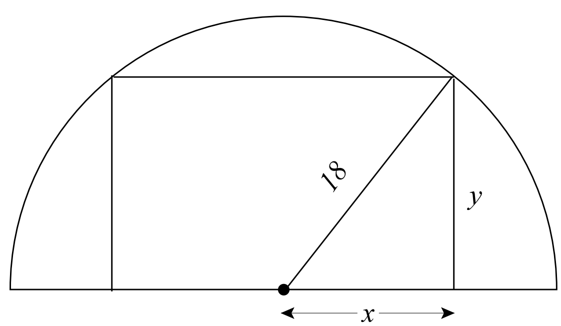
(b) Find the marginal revenue function.

(c) Find the marginal average profit function.

(d) Find the break-even points.

13. (10 Points) For Jobo Appliance Company, the monthly profit from the sale of x mixers and y blenders is given by $P(x, y) = 6xy - 2x^2 - 3y^3 + 3$ where x and y are measured in hundreds and P is measured in thousands of dollars. Find the number of mixers and blenders that should be sold to maximize the profit. What is the maximum profit?

Bonus Problem (10 Points) Find the dimensions of the rectangle of maximum area that can be cut out of a semicircle of radius 18 inches (see Figure).



ANSWER KEY TO SAMPLE EXAMINATION

1. (a) F (b) T (c) F (d) T (e) T

2. (a) $f'(x) = -9x^{-4} + \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}} - 5x^{-2}$

   (b) $\frac{53}{4}$

   (c) 2

   (d) 80

   (e) $r = 12$, $c(1,-2,3)$

3. (a) $f'(x) = (2x - 2x^{-3})(x^{\frac{1}{2}} + x^{-\frac{1}{2}}) + (x^2 + x^{-2})\left(\frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}\right)$

   (b) $f'(x) = \frac{1}{3}\left(\frac{x^2 + 3x}{2x - 3}\right)^{-\frac{2}{3}}\left(\frac{(2x+3)(2x-3) - (x^2+3x)(2)}{(2x-3)^2}\right)$

   (c) $f'(x) = -64(20x^2 - 4x + 10)^{-9}(40x - 4)$

4. (a) $f_{xx} = 72x^2 - 6y^3$

   (b) $f_{yy} = 10 - 18x^2y$

   (c) $f_{xy}(-1,-2) = 72$

5. Length = x, width = y, height = z and y = z

   S: Surface area = $2y^2 + 2xy$
   V: Volume = $xy^2 = 1024$

   $S(y) = 2y^2 + 2048y^{-1}$
   $S'(y) = 0$ gives $y = 8$

   $S''(8) > 0, y = 8$ (minima point)

   So $z = 8$, $x = 16$, $y = 8$

6. Let x represent the number of trees to be added. Then the yield function is:

$$y = (x + 20)(60 - 2x)$$

This gives (using maximizing techniques) $x = 5$.

Average yield per tree would be: 50 bushels
Total yield of the orchard is: 1500 bushels

7. Let length = x, width = y, height = z

$$V = 1400 = xyz$$

Cost function: $C(x,y,z) = 8xy + 6xy + 5(2yz + 2xz)$
$= 14xy + 10yz + 10xz$

$C(x,y) = 14xy + 14000(x^{-1} + y^{-1})$

Using partial derivatives for extrema problems, $C_x = 0$, $C_y = 0$ gives us the relation

$x^2y = 1000 = xy^2$. Thus, $x = y = 10$. So $z = 14$.

8. We must maximize the profit function $P(x,y,z) = 6x + 8y + 10z$, subject to constraint $x^2 + y^2 + z^2 = 800$.

So $F_x = 6 - 2\lambda x = 0$ (1)
$F_y = 8 - 2\lambda y = 0$ (2)
$F_z = 10 - 2\lambda z = 0$ (3)
$F_\lambda = x^2 + y^2 + z^2 - 800 = 0$ (4)

The first three equations give us $3y = 4x$, $4z = 5y$, $3z = 5x$. Now equation (4) becomes

$$x^2 + \frac{16x^2}{9} + \frac{25x^2}{9} = 800.$$

Thus, $x^2 = 144$ and $x = 12$. Therefore $y = 16$, $z = 20$.

9. (a) $y - 4 = -\frac{1}{3}(x + 1)$. It simplifies to: $x + 3y - 11 = 0$.

(b) $m = f'(x) = -2x$
$m = f'(1) = -2$

Tangent line is: $y - 3 = -2(x - 1)$.

10. (a) $B'(3) = 216$, $B(0) = 500$
(b) 5 days
(c) $B'(t) = 24t^2$

11. x-intercept = 2
y-intercept = 16
no symmentry on [0,3]
critical points at $x = 0$, $x = 2$
point of inflection at $x = 1.15$

f decreasing on [0,2]
f increasing on [2,3]
concave up on [1.15,3]
concave down on [0,1.15]
local extreme value P(2) = 0
local maximum value P(0) = 16
absolute maximum value P(3) = 25
absolute minimum value P(2) = 0
The graph is now simple to draw.

12. Average cost function:

$\bar{c}(x) = 2 + 56x^{-1}$

(a) $\bar{c}(8) = 9$ dollars.
(b) $R'(x) = 20 - 2x$.
(c) $\bar{P}'(x) = 56x^{-2} - 1$
(d) Breakeven points $x = 4$ and $x = 14$

13. $x = 150$ mixers, $y = 100$ blenders
Maximum profit: $4500.

Bonus Problem.
Maximize the area function $A(x) = 2xy = 2x(324 - x^2)^{\frac{1}{2}}$, and get $x = 9\sqrt{2}$ inches $y = 18\sqrt{2}$.